Mathematics of Computation

A Quarterly Journal
Edited by

XIV

NOS. 69-72

1960

Formerly: Mathematical Tables and other Aids to Computation

Published by the

National Academy of Sciences—National Research Council

Washington, D. C.

The SIAM Review

ARTICLES

On the role of professional societies in stimulating and guiding of research *R. F. Drenick*

Optimum allocation of discharge to units in a hydroelectric generating station . . . . . *B. Bernholtz*

Applications of graphs and boolean matrices to computer programming *Rosalind B. Marimont*

A note concerning orthogonal polynomials . . . . . *Ralph Hoyt Bacon*

Special block iterations with applications to Laplace and biharmonic difference equations . . . . . *Herbert B. Keller*

A theorem on determinants . . . . . *Eugen Gott*

Torpedo hit probabilities . . . . . *E. S. Wolk*

PROBLEMS

On a switching circuit . . . . . *Herbert A. Cohen*

A pie problem . . . . . *Larry Shepp*

A parking lot design . . . . . *D. J. Newman*

A minimum time path . . . . . *W. L. Bade*

A sorting problem . . . . . *Walter Weissblum*

BOOK REVIEWS

Approximate Methods of Higher Analysis (Kantorovich and Krylov) *T. J. Rivlin*

Handbook of Automation, Computation, and Control, Volume 2, Computers and Data Processing (Grabbe, Ramo, and Wooldridge) *Richard M. Karp*

NEWS AND NOTICES

the

January 1960 · Vol. 14, No. 69

Mathematics of Computation

A journal devoted to advances in numerical analysis, the application of computational methods, mathematical tables, high-speed calculators and other aids to computation

Formerly: Mathematical Tables and other Aids to Computation

Published Quarterly by the

National Academy of Sciences—National Research Council

Information to Subscribers

The journal is published quarterly in one volume per year with issues numbered serially since Volume I, Number 1. Starting with January, 1959 subscriptions are \$8.00 per year, single copies \$2.50. Back issues are available as follows:

- Volume I (1943–1945), Nos. 10 and 12 *only* are available; \$1.00 per issue.
- Volume II (1946–1947), Nos. 13, 14, 17, 18, 19, and 20 *only* available; \$1.00 per issue.
- Volume III (1948–1949), Nos. 21–28 available. \$4.00 per year (four issues), \$1.25 per issue.
- Volume IV (1950 through 1958), all issues available; \$5.00 per year, \$1.50 per issue.

Microcard Edition

Volumes I–X (1943–1956), Nos. 1–56 are now available on Microcards and may be purchased from the Microcard Foundation, Box 2145, Madison 5, Wisconsin, at a cost of \$20.00 for the complete set. Succeeding volumes are available on request.

Information to Contributors

All contributions intended for publication in *Mathematics of Computation* and all books for review should be addressed to H. Polachek, Technical Director, Applied Mathematics Laboratory, David Taylor Model Basin, Washington 7, D. C. The author may suggest an appropriate editor for his paper. Manuscripts should be typewritten double-spaced in the format used by the journal. Authors should submit the original and one copy, and should retain one copy.

Subscriptions, address changes, business communications and payments should be sent to:

THE PRINTING AND PUBLISHING OFFICE
THE NATIONAL ACADEMY OF SCIENCES
2101 Constitution Avenue
Washington 25, D. C.

PUBLISHED BY THE
NATIONAL ACADEMY OF SCIENCES—NATIONAL RESEARCH COUNCIL
Mt. Royal and Guilford Avenues, Baltimore 2, Md.
Second-class postage paid at Baltimore, Md.

el

h-
ce

n,

ed
re

00

s),

er

be
a

all
ed
he
be
b-

be

Em
Aic
1-2
S.O,

oth
ne
fle
in
As
se
jec
te
in
te

in
sc
qu
na
ve
pu
na
wh
th
tio
Pu
su
an

of
tr
in
tr
ph
ea
co
wh
st
sc
tio
sp

m
m
eq
di
ai

Engin.
Direct
1-26-60
S.O. 64586

8841A
New

MATHEMATICS OF COMPUTATION

Beginning with the January 1960 issue, the title of *Mathematical Tables and other Aids to Computation* has been changed to *Mathematics of Computation.* The new name was unanimously adopted by the Editorial Committee in order to reflect the broadened scope of the Journal, which has expanded to meet the need in this country for a publication devoted to numerical analysis and computation. As stated in the announcement of the October issue, the new title in no way represents a diminished interest in mathematical tables, which will continue to be a subject of major emphasis, as in the past. It recognizes, however, an increased interest in other areas in the field of *Mathematics of Computation*, which have grown in importance and in which rapid advances are being made in the present era of technological progress.

The first issue of *Mathematical Tables and other Aids to Computation* appeared in January 1943, more than one year prior to the completion of the first large scale computer—the Harvard Automatic Sequence Controlled Calculator. It is quite evident, however, that Professor R. C. Archibald, when he founded the journal, had to a large extent foreseen the great progress which was to follow in the development of instruments for computation. In the introductory remarks which he published in the first issue he established as one of the principal goals of the Journal, "to serve as a clearing-house for information" concerning tools of computation which have been "vastly multiplied during the past decade." He further speaks of these tools, or accounts of them, which "are to be found in an enormous international range of book, pamphlet, and periodical publication, not only in the fields of Pure Mathematics, Physics, Statistics, Astronomy, and Navigation, but also in such fields as Chemistry, Engineering, Geodesy, Geology, Physiology, Economics, and Psychology."

Indeed during the lifetime of this journal we have witnessed the development of one of the most promising technological tools ever devised by man—the electronic digital calculator. During this period the speed of digital calculators has increased one-hundred-thousand-fold. High-speed calculators are being used to translate from one language to another, to track satellites, to compose "symphonies," to design nuclear reactors, to forecast weather, to monitor this country's early warning system, to play chess, to simulate the motion of a submarine, or to compute fall-out patterns. This is but a small sample of the many complex tasks which are being performed by these devices. However, we are only in the initial stages of this development. It is not yet sixteen short years since the first large-scale automatic calculator has been placed in operation. *Mathematics of Computation* will in part be devoted to the exploration of new areas of applications for high-speed computer devices in every field of human endeavor.

Substantial progress has also been made in the development of numerical methods. For instance, until very recently it has not been feasible to obtain numerical solutions for most types of systems of ordinary and partial differential equations. This is no longer the case. Numerical solutions of complex systems of differential and integral equations—based on the methods of finite differences and aided by the availability of high-speed calculators—are now readily attainable.

This constitutes a major advance in applied mathematics. The mathematical equations which govern the solution of problems in many areas of engineering and theoretical physics are systems of differential equations. Supersonic and subsonic aerodynamics, nuclear reactor design theory, heat transfer, propagation of electromagnetic and acoustic waves are but a few illustrative fields which fall in this category. *Mathematics of Computation* will be devoted to advances in numerical analysis with the view toward further expansion of our capabilities to obtain numerical solutions to systems of mathematical relations of any type.

Many new mathematical disciplines are in the process of development which are required to advance the theory and foster the application of numerical methods. Information theory, the logic of automata, operational analysis, the theory of games, and linear programming are a few examples of such fields of current interest. These and many other mathematical developments will be required to comprehend clearly and utilize fully the vast potential of modern computational devices. *Mathematics of Computation* will be available for the publication of advances in modern mathematical theories related to computation.

In view of the phenomenally rapid progress in the development of computational devices and the equally impressive advances in numerical analysis, we look forward with keen interest and considerable expectation toward the future, to witness the exciting technological progress that will result from this major advance or break-through in applied mathematics. With this entire field of modern mathematics, embracing

(1) advances in numerical analysis,
(2) the application of numerical methods and high-speed calculator devices,
(3) the computation of mathematical tables,
(4) the development of new mathematical disciplines related to computation,
(5) the theory of high-speed calculating devices and other aids to computation,

we have associated the title,

Mathematics of Computation.

H. P.

Quadrature Formulas Involving Derivatives of the Integrand

By Preston C. Hammer and Howard H. Wicke

1. Introduction. In this paper we demonstrate the existence of quadrature formulas of the following types. For k an odd positive integer,

$$(1) \qquad \int_{-1}^{1} f(x)\,dx = 2 \sum_{i=0}^{(k-1)/2} \frac{f^{(2i)}(0)}{(2i+1)!} + \sum_{j=1}^{m} a_j\,[f^{(k)}(x_j) - f^{(k)}(-x_j)] + R_{m,k}(f).$$

For k an even positive integer,

$$(2) \qquad \int_{-1}^{1} f(x)\,dx = 2 \sum_{i=0}^{(k-2)/2} \frac{f^{(2i)}(0)}{(2i+1)!} + \sum_{j=1}^{m} a_j\,[f^{(k)}(x_j) + f^{(k)}(-x_j)] + R_{m,k}(f).$$

Each of formulas (1) and (2) will be shown to exist so that they are exact for polynomials of degree at most $4m + k$ for odd k, and $4m + k - 1$ for even k.

It is possible to use tables published by Hammer, Marlowe, and Stroud [2] and extended by Fishman [1] to obtain formulas of the form

$$(3) \qquad \int_{0}^{1} f(x)\,dx = f(1) + \sum_{j=1}^{m} a_j f'(x_j) + R_m(f)$$

and likewise formulas using higher derivatives. These formulas are asymmetrical and for some uses would be more appropriate than formula (1) or (2).

It is considered that formula (1), (2), or (3) may be useful when the integrand function is represented as a variable integral for which derivatives may be easier to compute than the integrand function itself. It is also anticipated that these formulas may be used in the numerical solution of differential equations.

Kopal [3] has devised formulas using the first derivative values. His approach leaves difficulties in establishing existence and reality of evaluation points and in some cases he has computed more than one formula of the same degree. The method we propose has direct connection with the established theory of orthogonal polynomials which gives the existence, reality, and distinctness of the evaluation points and the positiveness of the weights a_j. A linear transformation to give integration limits $-h$, h in formula (1) or (2) results in multiplying each derivative of order n by h^{n+1}.

We have not identified the orthogonal polynomial systems with any treated in detail in the literature. However, tables of the a_j and x_j for $k = 1, 2$, $m = 1$ (1) 10, and $k = 3, 4$, $m = 1$ (1) 9, have been computed by G. W. Struble in [5], where tables for the $\mu_{m,k}^{(m)}$ of the remainder terms are also to be found. For purposes of computation we give a standard type recursion formula permitting generation of each polynomial in a sequence from its two predecessors.

Throughout the paper we assume that the integrand function has all-order derivatives appearing and that these are continuous.

Received April 16, 1959; in revised form, July 9, 1959. This work is supported in part by the Office of Ordnance Research, U. S. Army.

2. Reduction of the Problems. It is well known and readily established that

$$\int_0^1 \left(\int_0^x\right)^n g(x)(dx)^{n+1} = \frac{1}{n!}\int_0^1 (1-x)^n g(x)\,dx, \tag{4}$$

where $\left(\int_0^x\right)^n g(x)(dx)^n$ denotes the result of integrating $g(x)$ successively n times over $(0, x)$. Now our formulas (1) and (2), by choice of form, will hold for every odd integrand function $f(x)$ and hence we write $f(x) = f_0(x) + f_1(x)$, where

$$f_0(x) = \frac{f(x) + f(-x)}{2} \text{ and } f_1(x) = \frac{f(x) - f(-x)}{2}.$$

The functions f_0 and f_1 may be called the *even* and *odd components* of f, respectively. We now observe

$$\int_{-1}^1 f(x)\,dx = 2\int_0^1 f_0(x)\,dx \text{ and } f_0^{(k)}(x) = \frac{f^{(k)}(x) + (-1)^k f^{(k)}(-x)}{2}. \tag{5}$$

Moreover,

$$\int_0^1 f_0(x)\,dx = \int_0^1 \left(\int_0^x\right)^k f_0^{(k)}(x)(dx)^{k+1} + S_k(f), \text{ where}$$

$$S_k(f) = \begin{cases} \displaystyle\sum_{i=0}^{(k-1)/2} \frac{f^{(2i)}(0)}{(2i+1)!}, & \text{for } k \text{ odd} \\ \displaystyle\sum_{i=0}^{(k-2)/2} \frac{f^{(2i)}(0)}{(2i+1)!}, & \text{for } k \text{ even}. \end{cases} \tag{6}$$

Hence our formulas (1) and (2) may be written, using formulas (4), (5), and (6), and dropping $2S_k(f)$ from each side, as follows:

$$\frac{2}{k!}\int_0^1 (1-x)^k f_0^{(k)}(x)\,dx = 2\sum_{j=1}^m a_j f_0^{(k)}(x_j) + R_{m,k}(f). \tag{7}$$

Now we only need derive formulas (7) exact for even-degree polynomials replacing $f_0(x)$. Then, if k is odd we have with $u = x^2$

$$\frac{2}{k!}\int_0^1 (1-x)^k x P(x^2)\,dx = \frac{1}{k!}\int_0^1 (1-\sqrt{u})^k P(u)\,du$$

$$= 2\sum_{j=1}^m a_j \sqrt{u_j} P(u_j), \text{ and thus,}$$

$$\frac{1}{k!}\int_0^1 (1-\sqrt{u})^k P(u)\,du = \sum_{j=1}^m b_j P(u_j), \qquad \text{where } b_j = 2a_j\sqrt{u_j}. \tag{8}$$

On the other hand, if k is even, so that $f_0^{(k)}(x)$ is replaced by an even polynomial $P(x^2)$, then we have with $u = x^2$, $b_j = 2a_j$,

$$\frac{1}{k!}\int_0^1 \frac{(1-\sqrt{u})^k}{\sqrt{u}} P(u)\,du = \sum_{j=1}^m b_j P(u_j). \tag{9}$$

Now formulas (8) and (9) are in standard form to apply the theory of orthogonal polynomials. The weight functions are positive-valued and appropriate. Hence there exists a sequence $\{P_{m,k}(u)\}$ of orthogonal polynomials for each $k = 1$,

2, $\cdots$, the zeros of which are the squares of the proposed evaluation points x_j, and positive numbers b_j from which the a_j are determined. The zeros are real, distinct, and in the interval.

The weight functions are simple in form but we have been unable to find references in which this class has been treated. For $k = 1$ we have verified that the system of polynomials is not a classical one since $\{P'_{m,1}(u)\}$ is not orthogonal. For purposes of calculating formulas, however, we state in the next section recursion formulas which permit sequential generation of the polynomials for each $k \geqq 1$.

3. Recursion Formula for Polynomials. The recursion formula stated here is of the standard sort, but its form is preferable to the one given by Szegö [4], p. 41, since it involves only the coefficients of P_{n-1} and P_{n-2} to calculate the multipliers B_n and C_n. This is important when explicit formulas for the multipliers have not been determined. We assume $\{P_n(u)\}$ is a system of polynomials with leading coefficient 1, orthogonal over the interval $[a, b]$ with respect to the weight function $w(u)$. Then we have

$$P_n(u) = (u + B_n)P_{n-1}(u) - C_nP_{n-2}(u). \qquad n \geqq 2 \tag{10}$$

If we define

$$\mu_n^{(k)} = \int_a^b w(u)u^kP_n(u)\,du \tag{11}$$

we may observe $\mu_n^{(k)} = 0$, $0 \leqq k < n$, and

$$\mu_n^{(n)} = \int_a^b w(u)u^nP_n(u)\,du = \int_a^b w(u)(P_n(u))^2\,du.$$

We also define $c_{n-1}^{(j)}$ as the coefficient of u^j in $P_{n-1}(u)$. Then B_n and C_n are given by

$$B_n = -\left(c_{n-1}^{(n-2)} + \frac{\mu_{n-1}^{(n)}}{\mu_{n-1}^{(n-1)}}\right) \tag{12}$$

$$C_n = \frac{\mu_{n-1}^{(n-1)}}{\mu_{n-2}^{(n-2)}}. \tag{13}$$

For our particular problem, the $\mu_{n-2}^{(n-2)}$, $\mu_{n-1}^{(n-1)}$, and $\mu_{n-1}^{(n)}$ are linear combinations of the coefficients of P_{n-2} and P_{n-1} with rational multipliers and the coefficients of every P_n are rational. Hence, in the absence of better formulas, starting with P_0 and P_1, we can generate the remaining polynomials, in principle. In practice, of course, the coefficients grow very rapidly in exact rational form.

Now $P_{0,k} = 1$ for every k, and, for degree one, $\{P_{1,k}(u)\}$ is given by

$$P_{1,2n-1}(u) = u - \frac{3}{(2n+3)(n+1)}, \qquad n = 1, 2, 3, \cdots, \tag{14}$$

and

$$P_{1,2n}(u) = u - \frac{1}{(2n+3)(n+1)}, \qquad n = 1, 2, 3, \cdots. \tag{15}$$

Hence, with the recursion formula (10), successive higher-degree polynomials may be calculated.

The *orthonormal* sequence $\{Q_n(u)\}$ is determined by

$$P_n(u) = \sqrt{\mu_n^{(n)}}\, Q_n(u). \tag{16}$$

4. Remainders for the Integration Formulas. In this section we give the remainder formulas. We will give the explicit functions to which Rolle's Theorem may be applied. The method is a variation of Markoff's for Hermite (osculating) interpolation (Szegö [4], p. 369.) As usual, we obtain the highest-degree polynomial $H(x)$ for which the formula is exact and which agrees with the integrand function or its derivatives at all evaluation points. Then we estimate

$$f^{(k)}(x) - H^{(k)}(x)$$

and $f_0^{(k)}(x) - H_0^{(k)}(x)$, and use formula (7) to find $R_{m,k}(f)$. (Here H_0 is the even component of H.)

For k *odd* and $2m + 1$ evaluation points $0, \pm x_1, \cdots, \pm x_m$, we require

$$H(0) = f(0), \qquad H^{(n)}(0) = f^{(n)}(0)$$

for $n = 1, \cdots, k$, $H^{(k)}(\pm x_j) = f^{(k)}(\pm x_j)$, $H^{(k+1)}(\pm x_j) = f^{(k+1)}(\pm x_j)$, $j = 1, \cdots, m$. Then $H(x)$ is a polynomial of degree at most $4m + k$. Let $P_m(x^2)$ be the polynomial $P_{m,k}(u)$ determining the evaluation points x_j by $P_m(x_j^2) = 0$, $j = 1, \cdots, m$. Then let x be any number which is none of $0, \pm x_j$. Define

$$F(z) = f^{(k)}(z) - H^{(k)}(z) - \frac{f^{(k)}(x) - H^{(k)}(x)}{x[P_m(x^2)]^2}\, z\,[P_m(z^2)]^2. \tag{17}$$

Then $F(z)$ vanishes at $x, 0, \pm x_j (2m + 2$ distinct points). Also, $F'(z)$ vanishes at $\pm x_j$ and $2m + 1$ other distinct points, or $4m + 1$ points. Hence, applying Rolle's Theorem there exists a ξ_1 such that $F^{(4m+1)}(\xi_1) = 0$, and thus

$$f^{(k)}(x) - H^{(k)}(x) = \frac{f^{(4m+k+1)}(\xi_1)\, x\,[P_m(x^2)]^2}{(4m + 1)!}. \tag{18}$$

From equation (18) applied to x and $-x$, and from the continuity of $f^{(4m+k+1)}(x)$, it follows that the even components satisfy

$$f_0^{(k)}(x) - H_0^{(k)}(x) = \frac{f^{(4m+k+1)}(\xi_2)\, x\,[P_m(x^2)]^2}{(4m + 1)!}. \tag{19}$$

Hence, from equation (7) by the continuity of $f^{(4m+k+1)}(x)$ and the first theorem of the mean for integrals, we have, with $u = x^2$, and for some $\eta \epsilon [-1, 1]$,

$$R_{m,k}(f) = \frac{f^{(4m+k+1)}(\eta)}{(4m + 1)!} \int_0^1 \frac{(1 - \sqrt{u})^k}{k!}\, [P_m(u)]^2\, du, \qquad \text{for } k \text{ odd}, \tag{20}$$

or replacing the integral by $\mu_{m,k}^{(m)}$,

$$R_{m,k}(f) = \frac{f^{(4m+k+1)}(\eta)}{(4m + 1)!}\, \mu_{m,k}^{(m)}, \qquad \text{for } k \text{ odd}. \tag{21}$$

For k even, we require $H(x)$, of at most degree $4m + k - 1$, such that

$$H(0) = f(0), \qquad H^{(n)}(0) = f^{(n)}(0),$$

for $n = 1, \cdots, k - 1$, $H^{(k)}(\pm x_j) = f^{(k)}(\pm x_j)$, $H^{(k+1)}(\pm x_j) = f^{(k+1)}(\pm x_j)$, for

for $j = 1, \cdots, m$. Then, for x not equal to one of $\pm x_j$, we define

$$F(z) = f^{(k)}(z) - H^{(k)}(z) - \frac{f^{(k)}(x) - H^{(k)}(x)}{[P_m(x^2)]^2}[P_m(z^2)]^2. \tag{22}$$

Now $F'(z)$ vanishes at $4m$ distinct points, and there exists a ξ_1 such that

$$F^{(4m)}(\xi_1) = 0,$$

which implies

$$f^{(k)}(x) - H^{(k)}(x) = \frac{f^{(4m+k)}(\xi_1)[P_m(x^2)]^2}{(4m)!}. \tag{23}$$

The even components f_0 and H_0 then satisfy

$$f_0^{(k)}(x) - H_0^{(k)}(x) = \frac{f^{(4m+k)}(\xi_2)[P_m(x^2)]^2}{(4m)!}, \tag{24}$$

using the same argument used in the passage from (18) to (19). Then, as before, substitution in equation (7) gives, with $u = x^2$, and for some $\eta\epsilon[-1, 1]$

$$R_{m,k}(f) = \frac{f^{(4m+k)}(\eta)}{(4m)!}\int_0^1 \frac{(1-\sqrt{u})^k}{k!\sqrt{u}}[P_m(u)]^2\,du, \qquad \text{for } k \text{ even}, \tag{25}$$

or, with the integral replaced by $\mu_{m,k}^{(m)}$,

$$R_{m,k}(f) = \frac{f^{(4m+k)}(\eta)}{(4m)!}\mu_{m,k}^{(m)}, \qquad \text{for } k \text{ even}. \tag{26}$$

The number $\mu_{m,k}^{(m)}$ may be determined simply in the low-degree cases by substituting in the numerical formula integrands x^{4m+k+1} if k is odd, and x^{4m+k} if k is even.

It may be noted that our explicit choice of H is made for convenience in each case. That is, formulas (1) and (2) hold with $f = H$ and $R_{mk} = 0$, and H is a highest-degree polynomial for which this is true—i.e., no higher even power of x may be included. Moreover, in formula (6) we have

$$\int_0^1 H_0(x)\,dx = \int_0^1\left(\int_0^x\right)^{k+1} H_0^{(k)}(x)(dx)^{k+1} + S_k(f), \tag{27}$$

since $f^{(j)}(0) = H^{(j)}(0)$, for appropriate values of j. Numerical tables of a_j, x, and the constant in the remainder have been calculated by G. W. Struble [5] for small values of k.

University of Wisconsin
Madison, Wisconsin and
Sandia Corporation
Albuquerque, New Mexico

1. H. Fishman, "Numerical integration constants," *MTAC*, v. 11, 1957, p. 1.
2. P. C. Hammer, O. J. Marlowe, & A. H. Stroud, "Numerical integration over simplexes and cones," *MTAC*, v. 10, 1956, p. 130–137.
3. Z. Kopal, *Numerical Analysis*, Wiley & Sons, New York, 1956.
4. G. Szegö, "Orthogonal polynomials," Amer. Math. Soc., *Colloquium Publs.*, v. 23, 1939.
5. G. W. Struble, "Quadrature formulas using values of derivatives," *Math. Comp.* (*MTAC*), v. 14, 1960, p. 8–12.

Tables for Use in Quadrature Formulas Involving Derivatives of the Integrand

By George Struble

1. Introduction. Tables of the a_j and x_j in the approximate quadrature formulas developed by Hammer and Wicke [1]

$$(1)\qquad \int_{-1}^{1} f(x)\,dx = 2\sum_{i=0}^{(k-1)/2} \frac{f^{(2i)}(0)}{(2i+1)!} + \sum_{j=1}^{m} a_j\,[f^{(k)}(x_j) - f^{(k)}(-x_j)],\ k \text{ odd}$$

$$(2)\qquad \int_{-1}^{1} f(x)\,dx = 2\sum_{i=0}^{(k-2)/2} \frac{f^{(2i)}(0)}{(2i+1)!} + \sum_{j=1}^{m} a_j\,[f^{(k)}(x_j) + f^{(k)}(-x_j)],\ k \text{ even}$$

are given in table 2 for $k = 1, 2$, $m = 1(1)10$. Coefficients of related orthogonal polynomials are given in table 1 for $k = 1, 2$, $m = 0(1)11$. The remainder terms for formulas (1) and (2) are

$$(3)\qquad R_{k,m} = \begin{cases} \dfrac{f^{(4m+k+1)}(\eta)}{(4m+1)!}\,C_{k,m},\ k \text{ odd} \\[2ex] \dfrac{f^{(4m+k)}(\eta)}{(4m)!}\,C_{k,m},\ k \text{ even}. \end{cases}$$

The $C_{k,m}$ are listed in table 3, for the same k and m as in table 2.

2. Calculations. The polynomials $P_{m,k}(x)$ were generated by orthonormalization of the sequence $1, x, \cdots, x^m$ with weight functions

$$(4)\qquad W_k(x) = \begin{cases} \dfrac{(1-\sqrt{x})^k}{k!},\ k \text{ odd} \\[2ex] \dfrac{(1-\sqrt{x})^k}{k!\sqrt{x}},\ k \text{ even} \end{cases}$$

on the interval $(0, 1)$. Their coefficients were obtained as solutions to systems of linear equations. The x_j are the square roots of the zeros r_j of $P_{m,k}(x)$. The numbers a_j are determined by

$$(5)\qquad a_j = \begin{cases} \dfrac{A_{m+1,k}}{2x_j\,P'_{m,k}(r_j)\,P_{m+1,k}(r_j)},\ k \text{ odd} \\[2ex] \dfrac{A_{m+1,k}}{2P'_{m,k}(r_j)\,P_{m+1,k}(r_j)},\ k \text{ even} \end{cases}$$

where $A_{m,k}$ is the quotient of leading coefficients of $P_{m,k}$ and $P_{m-1,k}$. The values of $C_{k,m}$ for use in the remainder terms were derived from

$$(6)\qquad C_{k,m} = \begin{cases} 2\left[\dfrac{(4m+1)!}{(4m+k+2)!} - \displaystyle\sum_{j=1}^{m} a_j\,x_j^{4m+1}\right],\ k \text{ odd} \\[2ex] 2\left[\dfrac{(4m)!}{(4m+k+1)!} - \displaystyle\sum_{j=1}^{m} a_j\,x_j^{4m}\right],\ k \text{ even,} \end{cases}$$

Received June 1, 1959; in revised form, August 18, 1959. The calculations and work were supported by the Wisconsin Alumni Research Foundation and the Office of Ordnance Research, U. S. Army.

Table 1

$k = 1$

$$P_0 = 1.73205\ 08075\ 6888$$

$$P_1 = 7.53370\ 80350\ 0884x - 2.26011\ 24105\ 0265$$

$$P_2 = 30.89988\ 23741\ 862x^2 - 23.66207\ 20883\ 408x + 2.68435\ 27159\ 0420$$

$$P_3 = 125.09567\ 55614\ 90x^3 - 156.54342\ 49462\ 16x^2 + 49.95110\ 87256\ 310x - 3.04662\ 58269\ 2541$$

$$P_4 = 503.92224\ 31999\ 51x^4 - 878.56845\ 42019\ 08x^3 + 480.77283\ 44538\ 39x^2 - 87.74164\ 92632\ 052x + 3.36803\ 13248\ 0860$$

$$P_5 = 2025.03058\ 73957\ 9x^5 - 4532.89143\ 04158\ 0x^4 + 3554.73528\ 18164\ 0x^3 - 1152.71421\ 91499\ 2x^2 + 138.22772\ 15120\ 70x - 3.65996\ 50721\ 1686$$

$$P_6 = 8126.60217\ 35502\ x^6 - 22225.89230\ 7314x^5 + 22794.37521\ 9445x^4 - 10831.68147\ 2854x^3 + 2371.48444\ 08556\ x^2 - 202.49633\ 19667\ 4x + 3.92935\ 16425\ 271$$

$$P_7 = 32585.34917\ 364x^7 - 1\ 05329.15543\ 67x^6 + 1\ 33708.35590\ 97x^5 - 84369.53936\ 860x^4 + 27562.57790\ 493x^3 - 4390.47486\ 2558x^2 + 281.55199\ 51626x - 4.18076\ 36516\ 86$$

$$P_8 = 1\ 30585.72574\ x^8 - 4\ 87145.05832\ x^7 + 7\ 37945.40041\ x^6 - 5\ 82772.09740\ x^5 + 2\ 56072.42057\ x^4 - 61782.59435\ 9x^3 + 7521.30300\ 46x^2 - 376.33313\ 306x + 4.41740\ 68709$$

$$P_9 = 5\ 23121.399x^9 - 22\ 12241.35x^8 + 38\ 96351.95x^7 - 37\ 02822.37x^6 + 20\ 55853.62x^5 - 6\ 74493.559x^4 + 1\ 25947.128x^3 - 12137.5185x^2 + 487.72367\ 9x - 4.63163\ 193$$

$$P_{10} = 20\ 95022.5x^{10} - 99\ 04599.4x^9 + 198\ 92582.x^8 - 221\ 24981.x^7 + 148\ 91824.x^6 - 62\ 25087.3x^5 + 15\ 95661.1x^4 - 2\ 38415.82x^3 + 18678.058x^2 - 616.56109x + 4.85521\ 90$$

$$P_{11} = 83\ 88194.x^{11} - 438\ 42170.x^{10} + 989\ 11790.x^9 - 1261\ 15400.x^8 + 999\ 31000.x^7 - 509\ 73540.x^6 + 167\ 70010.x^5 - 34\ 68084.x^4 + 4\ 25177.7x^3 - 27649.29x^2 + 763.6075x - 5.05931\ 2$$

$k = 2$

$$P_0 = 1.73205\ 08075\ 6888$$

$$P_1 = 12.70977\ 81860\ 449x - 1.27097\ 78186\ 0449$$

$$P_2 = 59.88757\ 23920\ 599x^2 - 29.17599\ 68063\ 882x + 1.20652\ 61837\ 2282$$

$$P_3 = 259.32961\ 55268\ 10x^3 - 242.98324\ 64964\ 47x^2 + 50.37043\ 53148\ 982x - 1.18192\ 23907\ 2002$$

$$P_4 = 1087.36971\ 21542\ 1x^4 - 1531.67966\ 14027\ 7\ x^3 + 633.73798\ 31134\ 97x^2 - 76.31728\ 62592\ 196x + 1.16909\ 13241\ 5856$$

$$P_5 = 4487.70266\ 55197\ 8x^5 - 8483.52105\ 08986\ 2x^4 + 5405.81028\ 77031\ 0x^3 - 1335.91407\ 05265\ 8x^2 + 106.99341\ 52181\ 20x - 1.16124\ 14205\ 1600$$

$$P_6 = 18355.66707\ 8423x^6 - 43644.30820\ 7345x^5 + 37942.62659\ 3113x^4 - 14721.14915\ 1884x^3 + 2472.29781\ 31355\ x^2 - 142.37405\ 03369\ 3x + 1.15594\ 79227\ 687$$

$$P_7 = 74659.95872\ 219x^7 - 2\ 14151.18872\ 09x^6 + 2\ 36965.50353\ 31x^5 - 1\ 27161.63787\ 63x^4 + 34058.68565\ 087x^3 - 4184.54516\ 3429x^2 + 182.43804\ 37977x - 1.15213\ 67777\ 11$$

$$P_8 = 3\ 02544.85398\ 6x^8 - 10\ 16908.12494\ x^7 + 13\ 69647.20094\ x^6 - 9\ 45055.96574\ 0x^5 + 3\ 54167.19148\ 2x^4 - 70308.79999\ 50\ x^3 + 6633.07482\ 984x^2 - 227.16794\ 4900x + 1.14926\ 09079\ 5$$

$$P_9 = 12\ 22819.3856x^9 - 47\ 14584.9219x^8 + 74\ 930684.8783x^7 - 63\ 42588.4460x^6 + 30\ 83282.8184x^5 - 8\ 64946.91266x^4 + 1\ 33312.13054x^3 - 9996.98758\ 11x^2 + 276.54933\ 828x - 1.14701\ 28163$$

$$P_{10} = 49\ 33033.914x^{10} - 214\ 63128.26x^9 + 393\ 48240.15\ x^8 - 395\ 46800.17x^7 + 237\ 46979.27x^6 - 87\ 07972.887x^5 + 19\ 12893.189x^4 - 2\ 36576.7122x^3 + 14474.00552\ x^2 - 330.57026\ 87x + 1.14520\ 6765$$

$$P_{11} = 198\ 72572.1x^{11} - 963\ 24095.7x^{10} + 2001\ 94335.x^9 - 2332\ 57625.x^8 + 1672\ 30868.x^7 - 762\ 13570.1x^6 + 220\ 33307.3x^5 - 39\ 12962.35x^4 + 3\ 98072.797x^3 - 20280.6219\ x^2 + 389.22470\ 8x - 1.14373\ 626$$

TABLE 2

m	j	k = 1 x_j	k = 1 a_j
1	1	.5477225575 05166	.3042903097 25092
2	1	.3721456511 38755	.2998694151 77678
	2	.7920059217 60865	.0695342880 47154
3	1	.2820900214 17742	.2625168155 42158
	2	.6268601608 87658	.1158516504 33743
	3	.8825310979 63249	.0226513358 72470
4	1	.2273288824 80793	.2283023164 62706
	2	.5140806061 43694	.1313690234 71769
	3	.7563681074 02448	.0510204966 76312
	4	.9248839673 29570	.0093443207 77566
5	1	.1904841804 26525	.2005148719 02275
	2	.4344049427 27343	.1330163257 25257
	3	.6535558761 15115	.0693217404 15712
	4	.8292980447 85322	.0254519669 74067
	5	.9479286287 01912	.0045107921 65497
6	1	.1639811690 9936	.1782134815 8969
	2	.3756528865 1806	.1290409387 9439
	3	.5724366400 9800	.0790426562 2466
	4	.7417866536 8859	.0393058004 7950
	5	.8740336979 4142	.0139874045 8116
	6	.9618129567 5531	.0024324030 3689
7	1	.1439912886 45	.1601387035 18
	2	.3307098113 32	.1229802576 16
	3	.5079816905 42	.0833391819 77
	4	.6669327097 06	.0490821180 75
	5	.8006064071 09	.0237352491 28
	6	.9033361608 21	.0082842932 84
	7	.9708112198 58	.0014235185 93
8	1	.1283703842	.1452769503
	2	.2952852684	.1163542537
	3	.4559661731	.0844453524
	4	.6038699343	.0553533943
	5	.7338622714	.0318100298
	6	.8415994425	.0151045544
	7	.9235314840	.0052043396
	8	.9769705854	.0008871299
9	1	.115823631	.132878407
	2	.266673730	.109816541
	3	.413291816	.083725810
	4	.550684038	.059056710
	5	.674964515	.037887335
	6	.782771237	.021412447
	7	.871235919	.010039513
	8	.938022434	.003428033
	9	.981369607	.000581047
10	1	.10552258	.12239571
	2	.24309589	.10363706
	3	.37773918	.08198747
	4	.50552941	.06098113
	5	.62345737	.04220687
	6	.72888457	.02667176
	7	.81949694	.01489473
	8	.89332330	.00691990
	9	.94876470	.00234732
	10	.98461980	.00039623

TABLE 2—*Continued*

m	j	$k = 2$	
		x_j	a_j
1	1	.3162277660 16838	.1666666666 66667
2	1	.2136036212 56443	.1437779500 85321
	2	.6644945298 23701	.0228887165 81346
3	1	.1638297723 39671	.1224797344 91539
	2	.5118494949 80572	.0395183540 10279
	3	.8050693913 12212	.0046685781 64849
4	1	.1336140770 13473	.1060360979 05865
	2	.4169305407 03968	.0469644870 94641
	3	.6739890075 18818	.0123633215 15102
	4	.8733068456 92833	.0013027601 51060
5	1	.1131072568 64736	.0933196598 68176
	2	.3520392516 58368	.0494739624 16079
	3	.5769763210 36054	.0188805340 81087
	4	.7683849647 09291	.0045407258 01483
	5	.9112358854 64930	.0004517844 99852
6	1	.0982014632 7176	.0832757144 1884
	2	.3048169927 4399	.0495578221 5891
	3	.5034255107 0724	.0234294328 9172
	4	.6810857911 4211	.0083180426 8343
	5	.8274681084 1897	.0019025149 7640
	6	.9344124389 0913	.0001831395 3766
7	1	.0868450485 199	.0751675980 885
	2	.2688827298 199	.0484760818 692
	3	.4461187708 020	.0263277958 602
	4	.6095094010 734	.0117357442 186
	5	.7517394338 839	.0039910960 716
	6	.8666985675 800	.0008849203 899
	7	.9495850305 493	.0000834301 748
8	1	.0778891768 89	.0684940996 00
	2	.2406041327 18	.0468585232 40
	3	.4003534653 41	.0280343978 67
	4	.5505429051 14	.0145005204 06
	5	.6858231067 44	.0062300015 19
	6	.8015939596 43	.0020600871 63
	7	.8940032842 24	.0004474372 29
	8	.9600499057 83	.0000415997 26
9	1	.0706369010 2	.0629090570 4
	2	.2177597386 8	.0450255142 4
	3	.3630238378 6	.0289225611 3
	4	.5014536828 7	.0166061935 8
	5	.6290237349 3	.0083212591 4
	6	.7422126043 9	.0034868683 1
	7	.8379675527 1	.0011308362 0
	8	.9137418873 3	.0002420894 9
	9	.9675684295 0	.0000222879 4
10	1	.0646394150	.0581677528
	2	.1989137370	.0431420181
	3	.3320234405	.0292595373
	4	.4601078372	.0181452843
	5	.5800614933	.0101353888
	6	.6891397684	.0049671598
	7	.7849102517	.0020445253
	8	.8652666620	.0006538257
	9	.9284600004	.0001385086
	10	.9731500562	.0000126607

TABLE 3*

$k = 1$		$k = 2$	
m	$C_{1,m}$	m	$C_{2,m}$
1	1.7619×10^{-2}	1	6.1905×10^{-3}
2	1.0473×10^{-3}	2	2.7882×10^{-4}
3	6.3902×10^{-5}	3	1.4869×10^{-5}
4	3.9380×10^{-6}	4	8.4576×10^{-7}
5	2.4386×10^{-7}	5	4.9654×10^{-8}
6	1.5142×10^{-8}	6	2.9680×10^{-9}
7	9.4185×10^{-10}	7	1.7940×10^{-10}
8	6.3112×10^{-11}	8	1.0922×10^{-11}
9	7.7×10^{-11}	9	6.8464×10^{-13}
10	-3.3×10^{-10}	10	-2.8×10^{-13}

* These figures include round-off errors.

and thus show effects of round-off errors in a_j and x_j as well as the theoretical remainder. Since the matrix of the linear system solved to find the coefficients of $P_{m,k}(x)$ is ill-conditioned, the accuracy of the figures decreases rapidly with increasing m. Figures are kept to the upper limit of accuracy. If results are rounded to one or two fewer significant figures than are carried in the table of x_j, there is no doubt of the accuracy of the digits kept.

The formulas were derived in [1]. The calculations were carried out in the Numerical Analysis Laboratory of the University of Wisconsin on the IBM 650. An 18-digit floating-decimal interpretive routine by Eugene Albright and a linear systems solver by Gerald Thorne were used. I am indebted to William Kammerer for several helpful suggestions.

University of Wisconsin
Madison, Wisconsin

1. P. C. HAMMER & H. H. WICKE, "Quadrature formulas involving derivatives of the integrand," *Math. Comp.* (*MTAC*), v. 14, 1960, p. 3–7.

Numerical Quadrature Over a Rectangular Domain in Two or More Dimensions

Part 1. Quadrature over a Square, Using up to Sixteen Equally Spaced Points

By J. C. P. Miller

1. Introduction. Except for a section of a paper by Bickley [1] which gives some of the results that follow, there seems to be very little in print concerning numerical quadrature over rectangular domains in two or more dimensions. This is perhaps because numerical evaluations can be readily made by using one-dimensional formulas on each variable in turn—such formulas as the Euler-Maclaurin formula, Gregory's formula, Simpson's rule, Newton's three-eighths rule, or Gauss's formulas, to name a few. This restriction to "products" of one-dimensional formulas limits the field unduly, and may lead to an excessively large amount of work. Thus, in n dimensions, use of Gauss's 3-point formula involves 3^n points, whereas comparable accuracy may be obtained with about $2n^2$ points.

In this first note we explore some of the simple possibilities corresponding to Simpson's rule and the three-eighths rule, applied to 9 or 16 points equally spaced over a square, the corner points being included.

2. Integration over a Square. We restrict the domain of integration to be a square; this covers any rectangular domain by change of scale. We take the center of the square as origin of coordinates, and in the first place take the side of the square to be $2h$; and the integral as

$$I = \int_{-h}^{h} \int_{-h}^{h} f(x, y)\, dx\, dy. \tag{2.1}$$

We assume that $f(x, y)$ can be expanded as far as we need in a Taylor series in x and y. In other words, we suppose that $f(x, y)$ can be represented adequately by a polynomial in x and y, with an error term which we shall suppose may be estimated by considering a few of the more significant neglected terms. We shall not give an accurate error analysis.

It is clear that polynomial terms involving an odd power of either variable will contribute nothing to the integral; we shall also group ordinates in sets such that the total contribution to their sum is zero for such terms involving an odd power of either variable. For example, the points $(h, 0)$, $(-h, 0)$, $(0, h)$, $(0, -h)$ form one such group. With this grouping we can then eliminate from consideration all terms of the Taylor expansion which do not involve an even power of both variables.

3. Method of Derivation. We seek approximate formulas of the form

$$I = \int_{-h}^{h} \int_{-h}^{h} f(x, y)\, dx\, dy \doteq \sum A_{r,s}(rh, sh), \tag{3.1}$$

Received September 18, 1959.

in which $A_{r,s} = A_{s,r} = A_{|r|,|s|}$, and which will be exact for appropriate polynomial functions $f(x, y)$ of sufficiently low degree.

Two approaches, not entirely distinct, may be used:

(i) We may give $f(x, y)$ special forms and choose coefficients $A_{r,s}$ to fit these exactly. Such forms are 1, x^2 (or the equivalent symmetrical form $x^2 + y^2$), x^4, x^2y^2, etc

(ii) We may expand $f(x, y)$ as a Taylor series and evaluate I both by means of the integral and by means of the sum, and equate coefficients.

We shall give an example of each approach, in order to bring out an interesting point concerning the proper choice of special forms.

4. Nine-point Formulas (i). Consider the nine points, grouped as indicated by the semi-colons: $(0, 0)$; $(h, 0)$, $(-h, 0)$, $(0, h)$, $(0, -h)$; (h, h), $(-h, h)$, $(h, -h)$, $(-h, -h)$; and the four special functions $f(x, y) = 1, x^2, x^4, x^2y^2$ with values at the nine points as follows

$$(4.1) \qquad \boxed{\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}} \qquad \boxed{\begin{matrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{matrix}} \qquad \boxed{\begin{matrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{matrix}} \qquad \boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{matrix}}$$

The values of $I/4h^2$ are

$$(4.2) \qquad 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{9}.$$

Hence, we wish to have

$$(4.3) \qquad \begin{cases} A_{0,0} + 4A_{1,0} + 4A_{1,1} = 1 \\ \qquad\quad 2A_{1,0} + 4A_{1,1} = \frac{1}{3} \\ \qquad\quad 2A_{1,0} + 4A_{1,1} = \frac{1}{5} \\ \qquad\qquad\qquad 4A_{1,1} = \frac{1}{9} \end{cases}$$

We note at once that two of the equations are incompatible. Since we have 4 equations for 3 unknowns, we can still solve them if we discard the third (since the correct result for x^2 must have precedence over the correct result for x^4). We obtain

$$A_{0,0} = \tfrac{4}{9} \qquad A_{1,0} = \tfrac{1}{9} \qquad A_{1,1} = \tfrac{1}{36}$$

which is precisely the Simpson's rule "product," see Bickley [1], eq. (22), with multipliers

$$(\mathrm{A}) \qquad \boxed{\begin{matrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{matrix}} \div 36$$

to give $I/4h^2$. The main error term clearly concerns the coefficients of the terms in x^4 and y^4 and is

$$\frac{h^4}{180}\left(\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4}\right)_{0,0}.$$

5. Nine-point Formulas (ii). We now consider the same problem by expanding and equating coefficients. We have, using $f_{r,s}$ for $f(rh, sh)$, and f_0 simply for $f(0, 0)$,

$$f(x, y) = f_0 + \frac{x^2}{2!}\frac{\partial^2 f_0}{\partial x^2} + \frac{y^2}{2!}\frac{\partial^2 f_0}{\partial y^2} + \frac{x^4}{4!}\frac{\partial^4 f_0}{\partial x^4} + \frac{6x^2y^2}{4!}\frac{\partial^4 f_0}{\partial x^2 \partial y^2} + \frac{y^4}{4!}\frac{\partial^4 f_0}{\partial y^4} + \cdots \tag{5.1}$$

whence

$$I = 4h^2\left[f_0 + \frac{h^2}{3!}\left(\frac{\partial^2 f_0}{\partial x^2} + \frac{\partial^2 f_0}{\partial y^2}\right) + \frac{h^4}{5!}\left(\frac{\partial^4 f_0}{\partial x^4} + \frac{10}{3}\frac{\partial^4 f_0}{\partial x^2 \partial y^2} + \frac{\partial^4 f_0}{\partial y^4}\right) + \cdots\right] \tag{5.2}$$

Following Bickley [1], we write this concisely in terms of

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{and} \quad \mathfrak{D}^4 \equiv \frac{\partial^4}{\partial x^2 \partial y^2}. \tag{5.3}$$

Then, adding further terms for later use,

$$\begin{aligned} I/4h^2 = f_0 &+ \frac{h^2}{3!}\nabla^2 f_0 + \frac{h^2}{5!}(\nabla^4 f_0 + \tfrac{4}{3}\mathfrak{D}^4 f_0) + \frac{h^6}{7!}(\nabla^6 f_0 + 4\mathfrak{D}^4\nabla^2 f_0) \\ &+ \frac{h^8}{9!}(\nabla^8 f_0 + 8\mathfrak{D}^4\nabla^4 f_0 + \tfrac{16}{5}\mathfrak{D}^8 f_0) + \cdots \end{aligned} \tag{5.4}$$

We have likewise

$$\begin{cases} f_{0,0} = f_0 \\ f_{1,0} + f_{0,1} + f_{-1,0} + f_{0,-1} = 4f_0 + \dfrac{2}{2!}h^2\nabla^2 f_0 + \dfrac{2}{4!}h^4(\nabla^4 f_0 - 2\mathfrak{D}^4 f_0) \\ \qquad + \dfrac{2}{6!}h^6(\nabla^6 f_0 - 3\mathfrak{D}^4\nabla^2 f_0) + \dfrac{2}{8!}h^8(\nabla^8 f_0 - 4\mathfrak{D}^4\nabla^4 f_0 + 2\mathfrak{D}^8 f_0) + \cdots \\ f_{1,1} + f_{-1,1} + f_{1,-1} + f_{-1,-1} = 4f_0 + \dfrac{4}{2!}h^2\nabla^2 f_0 + \dfrac{4}{4!}h^4(\nabla^4 f_0 + 4\mathfrak{D}^4 f_0) \\ \qquad + \dfrac{4}{6!}h^6(\nabla^6 f_0 + 12\mathfrak{D}^4\nabla^2 f_0) + \dfrac{4}{8!}h^8(\nabla^8 f_0 + 24\mathfrak{D}^4\nabla^4 f_0 + 16\mathfrak{D}^8 f_0) + \cdots \end{cases} \tag{5.5}$$

From these relations we obtain, for terms to h^4 in $I/4h^2$

$$\begin{cases} A_{0,0} + 4A_{1,0} + 4A_{1,1} = 1 \\ 2A_{1,0} + 4A_{1,1} = \tfrac{1}{3} \\ 2A_{1,0} + 4A_{1,1} = \tfrac{1}{5} \\ -4A_{1,0} + 16A_{1,1} = \tfrac{4}{15} \end{cases} \tag{5.6}$$

The first three equations are as in (4.3), and the second and third remain incompatible. Neglecting the third we can solve to give (see Bickley [1], eq. (20)) the multipliers for $I/4h^2$;

(B)

7	16	7
16	88	16
7	16	7

$\div 180$

The main error term is $\dfrac{h^4}{180}\nabla^4 f_0$.

The difference in the fourth equation between (4.3) and (5.6) leads to different formulas. The cause is reflected in the error term. In the second case we have ignored a term in $\nabla^4 f_0$ and removed the remaining term in $\mathfrak{D}^4 f_0$; in **4** we have ignored a term in

$$\left(\frac{\partial^4 f_0}{\partial x^4} + \frac{\partial^4 f_0}{\partial y^4}\right)$$

and removed the remaining term in $\mathfrak{D}^4 f_0$. The error term in the product-Simpson formula can be expressed alternatively in the form

$$\frac{h^4}{180}\,(\nabla^4 f_0 - 2\mathfrak{D}^4 f_0).$$

It is perhaps a moot point which formula will give better results, unless it is known that $f(x, y)$ is exactly, or almost a harmonic function. It is perhaps worth noting that for the method of **4** to yield the formula (B) it is necessary only to take a harmonic function instead of x^2y^2 in deriving the last equation. For example, $f(x, y) = x^4 - 6x^2y^2 + y^4$ gives values

$$\text{(5.7)} \qquad \boxed{\begin{matrix} -4 & 1 & -4 \\ 1 & 0 & 1 \\ -4 & 1 & -4 \end{matrix}} \qquad \text{and } I/4h^2 = -\tfrac{4}{15}h^4$$

and the fourth equation

$$\text{(5.8)} \qquad -4A_{1,0} + 16A_{1,1} = \tfrac{4}{15}.$$

6. Five-point and Other Formulas. The impossibility of removing all of the h^4 terms in the error in $I/4h^2$ by using 9 points, suggests the possibility of using fewer points, in fact five or eight, while still retaining an error of order h^4. There are three possibilities, using the first two equations only of (4.3) or (5.6).

(i) Take $A_{0,0} = 0$; this yields multipliers

$$\text{(C)} \qquad \boxed{\begin{matrix} -1 & 4 & -1 \\ 4 & 0 & 4 \\ -1 & 4 & -1 \end{matrix}} \div 12 \qquad \text{with main error } \tfrac{1}{180}h^4(\nabla^4 f_0 - 22\mathfrak{D}^4 f_0).$$

(ii) Take $A_{1,1} = 0$, giving

$$\text{(D)} \qquad \boxed{\begin{matrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{matrix}} \div 6 \qquad \text{with main error } \tfrac{1}{180}h^4(\nabla^4 f_0 - 7\mathfrak{D}^4 f_0).$$

(iii) Take $A_{1,0} = 0$, giving the "diagonal Simpson's rule"

$$\text{(E)} \qquad \boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 8 & 0 \\ 1 & 0 & 1 \end{matrix}} \div 12 \qquad \text{with main error } \tfrac{1}{180}h^4(\nabla^4 f_0 + 8\mathfrak{D}^4 f_0).$$

The first of these has little to commend it, with its large $\mathfrak{D}^4 f_0$ error term, and its negative multipliers. The other two (see Bickley [1], eqs. (23, 24)) are reasonably good. Over large areas, with many contiguous squares of area $4h^2$, they average respectively 3 and 2 points per square. With a square of side $6h$, for example, they give

$$\text{(F)}\qquad \boxed{\begin{array}{ccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}} \div 6 \qquad \text{and} \qquad \boxed{\begin{array}{ccccccc} 1 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 \\ 2 & 0 & 4 & 0 & 4 & 0 & 2 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 \\ 2 & 0 & 4 & 0 & 4 & 0 & 2 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 \\ 1 & 0 & 2 & 0 & 2 & 0 & 1 \end{array}} \div 12 .$$

The first of these has interior points with multipliers 2 (or zero) only. We note also that combination in proportions $\frac{8}{15}:\frac{7}{15}$ gives precisely the formula (B). One further formula is obtained by combining (D) and (E) in equal proportions:

$$\text{(G)}\qquad \boxed{\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 12 & 2 \\ 1 & 2 & 1 \end{array}} \div 24 \qquad \text{with main error } \tfrac{1}{180}h^4(\nabla^4 f_0 + \tfrac{1}{2}\mathfrak{D}^4 f_0)$$

and with simple multipliers and a good error term.

7. 16-point Formulas. These follow exactly the pattern for nine-point formulas; we merely quote results.

The method of **4**, with the same special functions, and with similar neglect of the incompatible third equation gives for $I/9h^2$, where

$$I = \int_{-3h/2}^{3h/2} \int_{-3h/2}^{3h/2} f(x, y)\, dx,\, dy,$$

the multipliers

$$\text{(A}'\text{)}\qquad \boxed{\begin{array}{cccc} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{array}} \div 64 \qquad \text{with main error term } \tfrac{1}{80}h^2(\nabla^4 f_0 - 2\mathfrak{D}^4 f_0).$$

The formula corresponding to (B) of **5** is

$$\text{(B}'\text{)}\qquad \boxed{\begin{array}{cccc} 7 & 13 & 13 & 7 \\ 13 & 47 & 47 & 13 \\ 13 & 47 & 47 & 13 \\ 7 & 13 & 13 & 7 \end{array}} \div 320 \qquad \text{with main error term } \tfrac{1}{80}h^4\nabla^4 f_0 .$$

8. Twelve-point Formulas and Others. As in **6** we can use the first two equations after setting one of the coefficients $A_{1,1}$, $A_{1,3}$, or $A_{3,3}$ equal to zero. This yields

twelve-point formulas

(i) with $A_{1,1} = 0$:

$$\text{(C}'\text{)} \qquad \boxed{\begin{array}{rrrr} -2 & 3 & 3 & -2 \\ 3 & 0 & 0 & 3 \\ 3 & 0 & 0 & 3 \\ -2 & 3 & 3 & -2 \end{array}} \div 16 \qquad \text{with main error term } \tfrac{1}{80}h^4(\nabla^4 f_0 - 47\mathfrak{D}^4 f_0)$$

(ii) with $A_{3,3} = 0$:

$$\text{(D}'\text{)} \qquad \boxed{\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{array}} \div 16 \qquad \text{with main error term } \tfrac{1}{80}h^3(\nabla^4 f_0 - 7\mathfrak{D}^4 f_0)$$

(iii) The "diagonal three-eighths rule", with $A_{1,3} = 0$:

$$\text{(E}'\text{)} \qquad \boxed{\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{array}} \div 16 \qquad \text{with main error term } \tfrac{1}{80}h^4(\nabla^4 f_0 + 13\mathfrak{D}^4 f_0).$$

Also, as in 6, we may combine the last two in the ratio $\frac{1}{3}:\frac{2}{3}$, to give

$$\text{(G}'\text{)} \qquad \boxed{\begin{array}{cccc} 1 & 2 & 2 & 1 \\ 2 & 7 & 7 & 2 \\ 2 & 7 & 7 & 2 \\ 1 & 2 & 2 & 1 \end{array}} \div 48 \qquad \text{with main error term } \tfrac{1}{80}h^4(\nabla^4 f_0 - \tfrac{1}{3}\mathfrak{D}^4 f_0).$$

Finally, we write out multipliers over a square of side $6h$ for the 12-point formulas (ii) and (iii) above (which average respectively 8 and 5 points per square of side $3h$)

$$\text{(F}'\text{)} \qquad \boxed{\begin{array}{ccccccc} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{array}} \div 16 \qquad \boxed{\begin{array}{ccccccc} 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 & 3 & 3 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 & 3 & 3 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 \end{array}} \div 16 .$$

Note once again the multipliers 2 or 0 only in the interior in the first case.

9. Numerical Illustrations and Comments. We consider the application of the formulas to two examples: (i) $f(x, y) = \cos x \cos y$; and (ii) $f(x, y) = \sin x \sinh y$, a harmonic function.

$$\text{(i) } I = \int_{-1}^{1}\int_{-1}^{1} \cos x \cos y \, dx \, dy = 4 \sin^2 1$$

whence $\qquad \frac{1}{4}I = 0.708073.$

The series (5.4) gives

f_0		1.000000
$\nabla^2 f_0/3!$	$-$	.333333
$\nabla^4 f_0/5!$	$+$	33333
$4\ \mathfrak{D}^4 f_0/3\cdot 5!$	$+$	11111
$\nabla^6 f_0/7!$	$-$	1587
$4\ \mathfrak{D}^4\nabla^2 f_0/7!$	$-$	1587
$\nabla^8 f_0/9!$	$+$	44
$8\ \mathfrak{D}^4\nabla^4 f_0/9!$	$+$	88
$16\ \mathfrak{D}^8 f_0/5\cdot 9!$	$+$	9
Sum		0.708078

This indicates the need for another term, and the alternation in sign, in this case, of terms with successive powers of h^2.

Table I shows the results of applying various formulas, with various values of h. Results are given to 5 decimals (based on calculations with 6-decimal function values). The column ϵ gives the error (formula—true value) and column C gives the computed value of the leading correction term, as listed in earlier paragraphs; ϵ and C are in units of the 5th decimal.

TABLE I

Formula	$h = 1$			$h = 1/2$			$h = 1/3$	
	Result	ϵ	C	Result	ϵ	C	Result	ϵ
(A)	0.71701	+894	−1111	0.70858	+51	−53	0.70817	+10
(B)	.72641	+1834	−2222	.70909	+102	−107		
(C)	.62309	−8498	+10000	.70345	−462	+481		
(D)	.69353	−1454	+1667	.70730	−77	+80	0.70792	−15
(E)	.76398	+5591	−6667	.71114	+307	−320		
(G)	.72876	+2069	−2500	.70922	+115	−120		

Formula	$h = 2/3$			$h = 1/3$		
	Result	ϵ	C	Result	ϵ	C
(A′)	0.71199	+392	−494	0.70830	+23	−24
(B′)	.71608	+801	−988	.70852	+45	−48
(C′)	.61988	−8819	+10617	.70319	−488	+511
(D′)	.70175	−632	+741	.70773	−34	+36
(E′)	.74269	+3462	−4198	.71000	+193	−202
(G′)	.71540	+733	−905	.70849	+42	−44

TABLE II

Formula	$h = 0.6$			Formula	$h = 0.4$		
	Result	ϵ	C		Result	ϵ	C
(A)	0.129414	+187	−186	(A′)	0.129310	+83	−83
(B)	.129227	0		(B′)	.129227	0	
(D)	.129879	+652	−652	(D′)	.129517	+290	−290
(E)	.128482	−745	+745	(E′)	.128689	−538	+538
(G)	.129181	−46	+47	(G′)	.129241	+14	−14

We note that the correction estimate C is numerically larger than the actual error ϵ in each case; this is due to the sign alternation mentioned earlier. We see also that, even with $h = 1$, C is quite a reasonable estimate of the size of ϵ.

Formulas (C), (C′), as expected, are not very good. Formulas (E), (E′), are also poor, since $\nabla^4 f_0$ and $\mathfrak{D}^4 f_0$ have the same sign. The "product-Simpson" rule gives the best results—best of all with $h = \frac{1}{3}$. Formulas (D) and (D′), as exhibited in (F), (F′) for a square of side $6h$, suggest that (D) gives better results than (D′) when the same points are used.

In practice, it is useful to use two formulas, and to compare results to give an estimate of the possible error. For this purpose (A) and (D) or (A′) and (D′) seem suitable pairs, unless it is known that f is harmonic or $\nabla^4 f_0$ is expected to be small compared with $\mathfrak{D}^4 f_0$.

(ii) $I = \int_0^{1.2} \int_0^{1.2} \sin x \sinh y \, dx \, dy = (1 - \cos 1.2)(\cosh 1.2 - 1) \doteqdot 0.516908.$

Approximations to $I/4 \doteqdot 0.1292271$ are listed; all 6 working decimals are shown in Table II. In this case, for a harmonic integrand, the superiority of (B) and (B′) is evident; (G′) is also good. In (B) and (B′), since $\nabla^2 f_0 = 0$, the error terms are of order h^8 instead of h^4. The quadrature of a harmonic function will be considered further in a later note.

I am glad to acknowledge help received with numerical calculations from Dr. J. W. Wrench, Jr. in Washington, D. C., and from W. R. Rosenkrantz at the University of Illinois; I am also grateful for facilities placed at my disposal at the Digital Computer Laboratory of the University of Illinois.

The University Mathematical Laboratory
Cambridge, England; and
The Digital Computer Laboratory
University of Illinois, Urbana, Illinois.

1. W. G. Bickley, "Finite difference formulae for the square lattice," *Quart. Jn. Mech. and Appl. Math.*, v. 1, 1948, p. 35–42.

Numerical Integration Formulas of Degree Two

By A. H. Stroud

1. Introduction. Here we discuss numerical integration formulas of the form

$$\int_R f(x)w(x)\,dx \cong \sum_i a_i f(\nu_i)$$

where R is a region in n-dimensional, real, euclidean space; $x = (x_1, x_2, \cdots, x_n)$; the a_i are constants; and the ν_i are points in the space. Most previous authors have given formulas for special regions (for a bibliography see [4]). Thacher [7] has given a method for constructing formulas of degree 2 with $n + 1$ points for general regions and of degree 3 with $2n$ points for certain symmetric regions; with his method, however, each region must also be treated separately. Our main results are to obtain specific formulas of degree 2 with $n + 1$ points for a general region satisfying a certain condition of non-degeneracy, and to show that for these regions such formulas cannot be obtained with fewer points. We also give a specific $2n$ point formula of degree 3 for a general centrally symmetric region. These results are a generalization of those of Georgiev [1, 2, 3] who has obtained similar results (but gives no general formulas) for $n = 2, 3$ with $w(x) \equiv 1$. Our results are obtained by a different method which was developed without knowledge of Georgiev's work.

2. Formulas of degree 2. We assume at first that an integration formula of degree 2 for R with respect to $w(x)$ can be obtained with $n + 1$ points

$$\nu_i = (\nu_{i1}, \cdots, \nu_{in}), \qquad i = 0, 1, \cdots, n.$$

Then the equations

$$\begin{aligned} &a_0 &&+ a_1 &&+ \cdots + a_n &&= c_0 \\ (1)\qquad &a_0\nu_{0j} &&+ a_1\nu_{1j} &&+ \cdots + a_n\nu_{nj} &&= c_{0j} \\ &a_0\nu_{0j}\nu_{0k} &&+ a_1\nu_{1j}\nu_{1k} &&+ \cdots + a_n\nu_{nj}\nu_{nk} &&= c_{jk} \qquad j, k = 1, 2, \cdots, n \end{aligned}$$

must be solved for both the a_i and the ν_i, where

$$c_0 = \int_R w(x)\,dx, \qquad c_{0j} = \int_R x_j\, w(x)\,dx, \qquad c_{jk} = \int_R x_j\, x_k\, w(x)\,dx.$$

We begin by writing (1) as the matrix equation

$$U^T A U = C \tag{2}$$

where

$$U = \begin{bmatrix} 1 & \nu_{01} & \cdots & \nu_{0n} \\ 1 & \nu_{11} & \cdots & \nu_{1n} \\ & & \cdots & \\ 1 & \nu_{n1} & \cdots & \nu_{nn} \end{bmatrix} \qquad A = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & a_n \end{bmatrix} \qquad C = \begin{bmatrix} c_0 & c_{01} & \cdots & c_{0n} \\ c_{01} & c_{11} & \cdots & c_{1n} \\ & & \cdots & \\ c_{0n} & c_{1n} & \cdots & c_{nn} \end{bmatrix}$$

and where we assume $0 < c_0 < \infty$ and $0 < |\det C| < \infty$.

Received March 18, 1958; in revised form August 4, 1959. This work was supported in part by the Office of Ordnance Research, U. S. Army and in part by the Wisconsin Alumni Research Foundation.

Since C is non-singular we can find a matrix T such that

$$T^{\mathsf{T}}U^{\mathsf{T}}AUT = T^{\mathsf{T}}CT = c_0E \tag{3}$$

where E is a diagonal matrix with elements ± 1. The method for finding T is well known (see [5], p. 56); we illustrate it using $n = 3$.

Since $c_0 \neq 0$ we define $t_{0i} = -c_{0i}/c_0$, $i = 1, 2, 3$, and form

$$T_1 = \begin{bmatrix} 1 & t_{01} & t_{02} & t_{03} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C_1 = T_1^{\mathsf{T}}CT_1 = \begin{bmatrix} c_0 & 0 & 0 & 0 \\ 0 & c_{11}^{(1)*} & c_{12}^{(1)} & c_{13}^{(1)} \\ 0 & c_{12}^{(1)} & c_{22}^{(1)} & c_{23}^{(1)} \\ 0 & c_{13}^{(1)} & c_{23}^{(1)} & c_{33}^{(1)} \end{bmatrix}.$$

Now if $c_{11}^{(1)*} = 0$ some $c_{1i}^{(1)} \neq 0$ since det $C \neq 0$. Assuming $c_{12}^{(1)} \neq 0$ we form

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & h & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C_2 = T_2^{\mathsf{T}}C_1T_2 = \begin{bmatrix} c_0 & 0 & 0 & 0 \\ 0 & 2hc_{12}^{(1)} + h^2c_{22}^{(1)} & c_{12}^{(1)} + hc_{22}^{(1)} & c_{13}^{(1)} + hc_{23}^{(1)} \\ 0 & c_{12}^{(1)} + hc_{22}^{(1)} & c_{22}^{(1)} & c_{23}^{(1)} \\ 0 & c_{13}^{(1)} + hc_{23}^{(1)} & c_{23}^{(1)} & c_{33}^{(1)} \end{bmatrix}$$

and choose h so that $c_{11}^{(1)} = 2hc_{12}^{(1)} + h^2c_{22}^{(1)} \neq 0$; if $c_{11}^{(1)*} \neq 0$ we take $h = 0$ so that $c_{11}^{(1)} = c_{11}^{(1)*}$. In this way we are assured that the element in the 1, 1 position is $\neq 0$.

Similarly we may find matrices T_3, T_4 and T_5 such that

$$C_3 = T_4^{\mathsf{T}}T_3^{\mathsf{T}}C_2\,T_3\,T_4 = \begin{bmatrix} c_0 & 0 & 0 & 0 \\ 0 & c_{11}^{(1)} & 0 & 0 \\ 0 & 0 & c_{22}^{(2)} & c_{23}^{(2)} \\ 0 & 0 & c_{23}^{(2)} & c_{33}^{(2)} \end{bmatrix} \qquad T_5^{\mathsf{T}}C_3\,T_5 = \begin{bmatrix} c_0 & 0 & 0 & 0 \\ 0 & c_{11}^{(1)} & 0 & 0 \\ 0 & 0 & c_{22}^{(2)} & 0 \\ 0 & 0 & 0 & c_{33}^{(3)} \end{bmatrix},$$

where $c_{22}^{(2)}$ and $c_{33}^{(3)}$ are $\neq 0$. Defining T_6 as the diagonal matrix

$$[1, [c_0/|\,c_{11}^{(1)}\,|]^{\frac12}, [c_0/|\,c_{22}^{(2)}\,|]^{\frac12}, [c_0/|\,c_{33}^{(3)}\,|]^{\frac12}]$$

we have finally $T = T_1T_2T_3T_4T_5T_6$.

We can assume E has the form $[1, 1, \cdots, 1, -1, \cdots, -1]$ since any other arrangement of $+1$'s and -1's can be put into this form by a suitable interchange of the rows of UT and the corresponding columns of $T^{\mathsf{T}}U^{\mathsf{T}}$. If C is positive definite (for example if $w(x)$ is of constant sign on R) E will be the identity. It should be noted that the first element of E will always be positive.

In the following we write

$$UT = \begin{bmatrix} 1 & \xi_{01} & \cdots & \xi_{0n} \\ 1 & \xi_{11} & \cdots & \xi_{1n} \\ & & \cdots & \\ 1 & \xi_{n1} & \cdots & \xi_{nn} \end{bmatrix} = \begin{bmatrix} 1 & \nu_{01} & \cdots & \nu_{0n} \\ 1 & \nu_{11} & \cdots & \nu_{1n} \\ & & \cdots & \\ 1 & \nu_{n1} & \cdots & \nu_{nn} \end{bmatrix} \begin{bmatrix} 1 & \tau_{01} & \cdots & \tau_{0n} \\ 0 & \tau_{11} & \cdots & \tau_{1n} \\ & & \cdots & \\ 0 & \tau_{n1} & \cdots & \tau_{nn} \end{bmatrix}.$$

Because UT is non-singular and $E^{-1} = E$ we easily obtain from (3)

$$(UT)E(UT)^{\mathsf{T}} = c_0A^{-1}.$$

In terms of the ξ_i this equation is

$$1 + \xi_{i1}\,\xi_{j1} + \cdots + \xi_{ip}\,\xi_{jp} - \xi_{i,p+1}\,\xi_{j,p+1} - \cdots - \xi_{in}\,\xi_{jn} = \frac{c_0}{a_1}\,\delta_{ij} \tag{4}$$

$$i, j, = 0, 1, \cdots, n.$$

where $p + 1$, $0 \leqq p \leqq n$, is the number of $+1$'s in E. We discuss the solution of (4); the ν_i are obtained from the ξ_i by $\nu_{ij} = \tau'_{0j} + \xi_{i1}\tau'_{1j} + \cdots + \xi_{in}\tau'_{nj}$, $i = 0, 1, \cdots, n$, $j = 1, \cdots, n$, where

$$T^{-1} = \begin{bmatrix} 1 & \tau'_{01} & \cdots & \tau'_{0n} \\ 0 & \tau'_{11} & \cdots & \tau'_{1n} \\ & & \cdots & \\ 0 & \tau'_{n1} & \cdots & \tau'_{nn} \end{bmatrix}.$$

We are only interested in real solutions of (1) and therefore precisely $n - p + 1$ of the a_i must be negative by Sylvester's "law of inertia" ([5], p. 56). If E is the identity ($p = n$) clearly we must have $0 < a_i < c_0$; if $p < n$ the only condition for the a_i is that they be non-zero.

Table 1 gives a particular solution of (4); we have assumed $a_0, \cdots, a_{n-p}$ negative and $a_{n-p+1}, \cdots, a_n$ positive. In the places where a double sign occurs we mean to use the lower sign for the last $n - p$ components of each vector and the upper sign for the first p components. Each ξ_i is real.

TABLE 1

$$\xi_0 = \left(0, 0, \cdots, 0, 0, \left[\frac{c_0 - a_0}{\pm a_0}\right]^{1/2}\right)$$

$$\xi_1 = \left(0, 0, \cdots, 0, \left[\frac{c_0(c_0 - a_0 - a_1)}{\pm(c_0 - a_0)a_1}\right]^{1/2}, \mp\left[\frac{\pm a_0}{c_0 - a_0}\right]^{1/2}\right)$$

$$\xi_2 = \left(0, 0, \cdots, \left[\frac{c_0(c_0 - a_0 - a_1 - a_2)}{\pm(c_0 - a_0 - a_1)a_2}\right]^{1/2}, \mp\left[\frac{\pm c_0 a_1}{(c_0 - a_0)(c_0 - a_0 - a_1)}\right]^{1/2}, \mp\left[\frac{\pm a_0}{c_0 - a_0}\right]^{1/2}\right)$$

$$\cdots\cdots\cdots$$

$$\xi_{n-2} = \left(0, \left[\frac{\pm c_0(c_0 - a_0 - \cdots - a_{n-2})}{(c_0 - a_0 - \cdots - a_{n-3})a_{n-2}}\right]^{1/2}, \cdots\right.$$
$$\left.\cdots, \mp\left[\frac{\pm c_0 a_2}{(c_0 - a_0 - a_1)(c_0 - a_0 - a_1 - a_2)}\right]^{1/2}, \left[\frac{\pm c_0 a_1}{(c_0 - a_0)(c_0 - a_0 - a_1)}\right]^{1/2}, \mp\left[\frac{\pm a_0}{c_0 - a_0}\right]^{1/2}\right)$$

$$\xi_{n-1} = \left(\left[\frac{\pm c_0(c_0 - a_0 - \cdots - a_{n-1})}{(c_0 - a_0 - \cdots - a_{n-2})a_{n-1}}\right]^{1/2}, \mp\left[\frac{\pm c_0 a_{n-2}}{(c_0 - a_0 - \cdots - a_{n-3})(c_0 - a_0 - \cdots - a_{n-2})}\right]^{1/2}, \cdots\right.$$
$$\left.\cdots, \mp\left[\frac{\pm c_0 a_2}{(c_0 - a_0 - a_1)(c_0 - a_0 - a_1 - a_2)}\right]^{1/2}, \mp\left[\frac{\pm c_0 a_1}{(c_0 - a_0)(c_0 - a_0 - a_1)}\right]^{1/2}, \mp\left[\frac{\pm a_0}{c_0 - a_0}\right]^{1/2}\right)$$

$$\xi_n = \left(\mp\left[\frac{\pm c_0 a_{n-1}}{(c_0 - a_0 - \cdots - a_{n-2})a_n}\right]^{1/2}, \mp\left[\frac{\pm c_0 a_{n-2}}{(c_0 - a_0 - \cdots - a_{n-3})(c_0 - a_0 - \cdots - a_{n-2})}\right]^{1/2}, \cdots\right.$$
$$\left.\cdots, \mp\left[\frac{\pm c_0 a_2}{(c_0 - a_0 - a_1)(c_0 - a_0 - a_1 - a_2)}\right]^{1/2}, \mp\left[\frac{\pm c_0 a_1}{(c_0 - a_0)(c_0 - a_0 - a_1)}\right]^{1/2}, \mp\left[\frac{\pm a_0}{c_0 - a_0}\right]^{1/2}\right)$$

From a particular solution ξ_{ij} of (4) other solutions may be obtained as follows. If

$$S = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \sigma_{11} & \cdots & \sigma_{1n} \\ & & \cdots & \\ 0 & \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$$

is a cogredient automorph of E, that is if $SES^T = E$, then

$$\xi'_{ij} = \xi_{i1}\sigma_{1j} + \cdots + \xi_{in}\sigma_{nj}$$

is also a solution. If Q is an arbitrary skew matrix of order $n + 1$, with first row and column entirely zero, such that $\det (E + Q)(E - Q) \neq 0$, then

$$S = (E + Q)^{-1}(E - Q)$$

is a cogredient automorph of E (see [5], p. 65) of the above form. If E is the identity S is orthogonal. We remark that in this latter case (4) determines the distances $d(\xi_i, 0)$ and $d(\xi_i, \xi_j)$, $i, j = 0, 1, \cdots, n$, $i \neq j$,

$$d(\xi_i, 0) = [(c_0 - a_i)/a_i]^{\frac{1}{2}} \qquad d(\xi_i, \xi_j) = [c_0(a_i + a_j)/a_i a_j]^{\frac{1}{2}}$$

The formulas discussed above are minimal; that is, similar formulas cannot be obtained with fewer points. For if a formula could be obtained with $m + 1$ points ν_i, $i = 0, 1, \cdots, m$, $m < n$, then equation (2) would still hold, where C is the same as before and

$$U = \begin{bmatrix} 1 & \nu_{01} & \cdots & \nu_{0n} \\ 1 & \nu_{11} & \cdots & \nu_{1n} \\ & & \cdots & \\ 1 & \nu_{m1} & \cdots & \nu_{mn} \end{bmatrix} \qquad A = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & a_m \end{bmatrix};$$

that is, U is a rectangular matrix. Since U and A have rank at most $m + 1$, then $U^T AU$ has rank at most $m + 1$ and therefore $\det (U^T AU) = 0$. By assumption $\det C \neq 0$ and thus (2) cannot hold for $m < n$.

3. Formulas of degree 3 for centrally symmetric regions. We assume R to be centrally symmetric with respect to the origin; then if x is in R, $-x$ is also in R. Let us further assume $w(-x) = w(x)$ for x in R. Then

$$\int_R x_i w(x)\, dx = \int_R x_i x_j x_k w(x)\, dx = 0, \qquad i, j, k = 1, \cdots, n.$$

We may obtain an integration formula of degree 3 for R with respect to $w(x)$ with $2n$ points as follows. Take the points to be ν_i, $-\nu_i$, $i = 1, \cdots, n$, and take ν_k, $-\nu_k$ to have common weight a_k. Any $2n$ points chosen in this way integrate exactly the monomials x_i, $x_i x_j x_k$ with respect to $w(x)$ over R. In addition we must solve

$$\begin{aligned} a_1 \qquad + a_2 \qquad + \cdots + a_n \qquad &= \tfrac{1}{2}c_0 \\ a_1\nu_{1j}\nu_{1k} + a_2\nu_{2j}\nu_{2k} + \cdots + a_n\nu_{nj}\nu_{nk} &= \tfrac{1}{2}c_{jk} \qquad j, k = 1, \cdots, n. \end{aligned}$$

The second of these may be written as the matrix equation (2) where now

$$U = \begin{bmatrix} \nu_{11} & \nu_{12} & \cdots & \nu_{1n} \\ \nu_{21} & \nu_{22} & \cdots & \nu_{2n} \\ & \cdots & & \\ \nu_{n1} & \nu_{n2} & \cdots & \nu_{nn} \end{bmatrix} \quad A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & a_n \end{bmatrix} \quad C = \frac{1}{2}\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{12} & c_{22} & \cdots & c_{2n} \\ & & \cdots & \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

and where we assume $-\infty < c_0 < \infty$ and $0 < |\det C| < \infty$.

We solve this equation by a method similar to that of the preceding section. We find a non-singular matrix T such that

$$T^{\mathsf{T}} U^{\mathsf{T}} A U T = T^{\mathsf{T}} C T = E$$

where E is diagonal with elements ± 1. Again it is convenient to assume

$$E = [1, \cdots, 1, -1, \cdots, -1]$$

where the first p elements are $+1$, $0 \leqq p \leqq n$. Now writing

$$UT = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ & \cdots & & \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{bmatrix} = \begin{bmatrix} \nu_{11} & \nu_{12} & \cdots & \nu_{1n} \\ \nu_{21} & \nu_{22} & \cdots & \nu_{2n} \\ & & \cdots & \\ \nu_{n1} & \nu_{n2} & \cdots & \nu_{nn} \end{bmatrix} \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1n} \\ \tau_{21} & \tau_{22} & \cdots & \tau_{2n} \\ & & \cdots & \\ \tau_{n1} & \tau_{n2} & \cdots & \tau_{nn} \end{bmatrix}$$

the ξ_{ij} may be solved for in terms of the a_i. This gives

$$(5) \qquad \xi_{i1}\xi_{j1} + \cdots + \xi_{ip}\xi_{jp} - \xi_{i,p+1}\xi_{j,p+1} - \cdots - \xi_{in}\xi_{jn} = \frac{1}{a_i}\delta_{ij} \qquad i, j = 1, \cdots, n$$

precisely $n - p$ of the a_i must be negative in order that the ξ_i be real.

If $a_1, \cdots, a_p$ are positive and $a_{p+1}, \cdots, a_n$ negative a particular solution of (5) is

$$\xi_i = (0, \cdots, 0, \sqrt{1/|a_i|}, 0, \cdots, 0) \qquad i = 1, \cdots, n$$

where the ith component of ξ_i is non-zero. If $S = (\sigma_{ij})$ is any cogredient automorph of E then $\xi'_{ij} = \xi_{i1}\sigma_{1j} + \cdots + \xi_{in}\sigma_{nj}$ is also a solution of (5). If E is the identity, that is, C is positive-definite, the solutions of (5) correspond to the sets of n orthogonal vectors in the space having the property that the ith vector of each set is a distance $\sqrt{1/a_i}$ from the origin.

4. Concluding remarks. The importance of the result given in this paper for formulas of degree 2 is that it is the first result (other than the trivial one point formula, the centroid of R, which integrates any linear function) which holds for an arbitrary region in n-dimensional space and which gives all such formulas containing the minimum number of points.

A question, which may have some practical importance, which may be asked about the above formulas of degree 2 concerns the conditions R must satisfy, say for $w(x) \equiv 1$, in order that such a formula will exist with all of its points interior to R. For example, can a formula interior to R be found if R is convex? if R is star-like about its centroid?

The error bound of von Mises [6] for n-dimensional integration formulas is very well suited for use with the formulas developed in this paper. In a later paper we will give specific values of this error bound for various known formulas.

I am especially indebted to Dr. P. C. Hammer for many discussions concerning this subject.

University of Wisconsin
Madison, Wisconsin

1. G. Georgiev, "Formulas of mechanical quadrature with minimal numbers of terms for multiple integrals," *Doklady Akad. Nauk. SSSR*, v. 83, 1952, p. 521–524. (Russian)
2. G. Georgiev, "Formulas of mechanical quadrature with equal coefficients for multiple integrals," *Doklady Akad. Nauk SSSR*, v. 89, 1953, p. 389–392. (Russian)
3. G. Georgiev, "Formulas of mechanical cubature with minimal numbers of terms," *Rozprawy Matematyczne*, No. 8, 1955, p. 72. (Russian; English summary)
4. P. C. Hammer & A. H. Stroud, "Numerical evaluation of multiple integrals II," *MTAC*, v. 12, 1958, p. 272–280.
5. C. C. MacDuffee, *Theory of Matrices*, Chelsea, New York, 1946.
6. R. von Mises, "Numerische Berechnung mehrdimensionaler Integrale," *Zeit. angew. Math. Mech.*, v. 34, 1954, p. 201–210.
7. H. C. Thacher, Jr., "Optimum quadrature formulas in s dimensions," *MTAC*, v. 11, 1957, p. 189–194.

Calculation of Transient Motion of Submerged Cables

By Thomas S. Walton and Harry Polachek

Abstract. The system of nonlinear partial differential equations governing the transient motion of a cable immersed in a fluid is solved by finite difference methods. This problem may be considered a generalization of the classical vibrating string problem in the following respects: a) the motion is two dimensional, b) large displacements are permitted, c) forces due to the weight of the cable, buoyancy, drag and virtual inertia of the medium are included, and d) the properties of the cable need not be uniform. The numerical solution of this system of equations presents a number of interesting mathematical problems related to: a) the nonlinear nature of the equations, b) the determination of a stable numerical procedure, and c) the determination of an effective computational method. The solution of this problem is of practical significance in the calculation of the transient forces acting on mooring and towing lines which are subjected to arbitrarily prescribed motions.

1. Introduction. This problem arose as a result of an urgent requirement by the Navy in connection with a series of nuclear explosion tests which were conducted in the Pacific. In preparation for these tests a number of ships were instrumented and moored at specified locations from the explosion point. These positions had to be maintained intact during the period preceding the explosion. However, the bobbing up and down of the ships due to ocean waves could excite transient forces in the mooring lines sufficient to break them and thus result in the loss of information from the tests. Several months prior to these tests a request was made to the Applied Mathematics Laboratory to calculate the magnitude of the forces acting on the mooring lines for waves of varying amplitude and frequency. The two factors which made a theoretical solution feasible at this time, whereas it would not have been possible several years ago, were: a) the availability of a high-speed computer and b) the recent progress made in the understanding and development of numerical methods for the solution of systems of partial differential equations by finite-difference methods.

Although this problem was solved to satisfy a specific request, it is more useful to regard it as the general problem of the two-dimensional motion of a cable or rope immersed in a fluid, and it becomes immediately apparent that its solution is applicable to a wide class of engineering problems involving the motion of cables, such as: a) the laying of submarine telegraph cables, b) the towing of a ship or other object in water, or c) the snapping of power lines as a result of transient forces caused by storms. The problem may be stated abstractly as follows: Given the initial conditions (i.e., position and velocity at any time, t_0) and boundary conditions (positions of end points at all times) of a cable immersed in a fluid, determine its subsequent motions. The motions are assumed to take place in two dimensions.

Received September 24, 1959. This work was done at the Applied Mathematics Laboratory, David Taylor Model Basin, and the paper is a condensation of DTMB Report 1279 [1].

Forces that are assumed acting on the cable are: a) forced motion of the extremities of the cable, b) damping or drag as it moves through the fluid, c) inertial reaction of the surrounding fluid, d) weight of the cable, and e) buoyancy. Variations in the mass as well as other physical properties of the cable along its length are allowed. However, in the present solution it is assumed that the cable is inextensible. The displacements may be large and the motions rapid, provided that all significant components of the driving motion lie in a frequency range well below the lowest natural frequency of the line for elastic (longitudinal) vibrations. In other words, the cable must be sufficiently short (or the velocity of propagation of elastic waves sufficiently great) that the line may be considered to be in equilibrium as far as longitudinal waves are concerned. In subsequent work the authors have carried out solutions for cables with elastic properties.

2. Derivation of Equations of Motion. The problem under consideration is a generalization of the classical problem of the motion of a vibrating string. We wish to deduce the approximate motion of a flexible steel cable without becoming involved in the explicit computation of the elastic forces which act on the cable. However, the formulation of the problem will be more general in several respects, namely:

a) Longitudinal as well as transverse motions of the line must be taken into consideration.

b) The occurrence of large displacements from the equilibrium configuration of the line must be permitted.

c) The extremities of the cable may be at different levels with the cable sagging between the positions of support. This requires that the weight of the line be taken into account. Thus, even when the line is in static equilibrium, the tension will not be uniform nor will the line be straight.

d) Since the cable is submerged, the static forces must include the buoyancy of the medium, and the dynamic forces must allow for the virtual inertia of the medium. Furthermore, it is necessary to make provision for damping forces due to the drag on the line whenever transverse motion is occurring.

e) Finally, it is desired to suspend concentrated loads at one or more points along the line and to change the diameter and linear density of the cable at specified points.

The best approach to the solution of a problem with such general specifications appears to be a numerical method based on finite-difference approximations. Inasmuch as we are committing ourselves to the eventual use of differences in both the time and space dimensions, it will be simpler to introduce the spacewise discreteness into the original formulation of the problem. We therefore proceed at once to the derivation of the equations of motion of a simplified model in which the distributed mass of the cable has been replaced by a series of discrete masses m_j attached to a weightless, inextensible line. This leads to a system of ordinary differential equations. It may be shown that, in the limit, the resulting equations pass over into the corresponding partial differential equations for the motion of a submerged cable.

Figure 1 shows a typical configuration of the system with the cable attached to a float at the surface and anchored to the bottom. Also, a heavy load is suspended

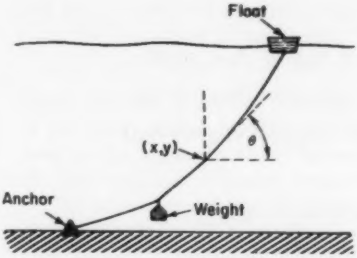

FIG. 1. —Mooring line.

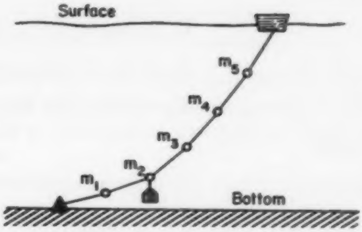

FIG. 2. —Discrete representation.

from a point near one end of the cable. Other boundary conditions are possible, but the equations of motion will be the same in any case. The horizontal and vertical coordinates of a point on the line are called x and y, respectively, and the angle between the horizontal and the tangent to the line is designated by θ. Figure 2 illustrates the corresponding discrete model for which the equations will actually be derived. The line is divided into segments in such a way that there will always be an integral number of them between any points where an abrupt change in some parameter occurs. The junctions between the segments are numbered according to the subscript index j, which runs from 0 at the anchor to s at the surface.

Before we can properly invoke Newton's law of motion, it is necessary to consider the inertial properties of the fluid in which the cable is immersed. We shall assume that the kinetic energy imparted to the surrounding medium is independent of the component of velocity parallel to the line, whereas it varies as the square of the component of velocity at right angles to the line. Thus, when an element of the cable is accelerated longitudinally, no hydrodynamic reaction occurs, but when the cable is accelerated transversely, it behaves as though it possessed additional inertia. For convenience in formulating the equations of motion in Cartesian coordinates, the transverse component of acceleration, namely, $-\ddot{x}\sin\theta + \ddot{y}\cos\theta$, can be further resolved into horizontal and vertical components (to which the corresponding components of the accompanying inertial reaction will be proportional). That is,

$$\text{Horizontal component} = \ddot{x}\sin^2\theta - \ddot{y}\sin\theta\cos\theta$$

$$\text{Vertical component} \quad = \ddot{y}\cos^2\theta - \ddot{x}\sin\theta\cos\theta$$

Thus, each component of the hydrodynamic reaction depends on both components of acceleration. In general, the reaction force is not parallel to the acceleration vector (except when the tangential component is zero), so that it is necessary to regard the inertial parameters of the system as tensors rather than simple scalars.

The differential equations governing the motion of the jth station on the line (see Fig. 2) can be written in matrix notation as follows:

$$\begin{pmatrix} I_j & -K_j \\ -K_j & J_j \end{pmatrix} \begin{pmatrix} \ddot{x}_j \\ \ddot{y}_j \end{pmatrix} = \begin{pmatrix} F_j^X \\ F_j^Y \end{pmatrix} \tag{2.1}$$

where

$$I_j = m_j + \tfrac{1}{2}(e_{j+\frac{1}{2}} \sin^2 \theta_{j+\frac{1}{2}} + e_{j-\frac{1}{2}} \sin^2 \theta_{j-\frac{1}{2}}) + m_j^X$$
$$J_j = m_j + \tfrac{1}{2}(e_{j+\frac{1}{2}} \cos^2 \theta_{j+\frac{1}{2}} + e_{j-\frac{1}{2}} \cos^2 \theta_{j-\frac{1}{2}}) + m_j^Y$$
$$K_j = \tfrac{1}{2}(e_{j+\frac{1}{2}} \sin \theta_{j+\frac{1}{2}} \cos \theta_{j+\frac{1}{2}} + e_{j-\frac{1}{2}} \sin \theta_{j-\frac{1}{2}} \cos \theta_{j-\frac{1}{2}})$$

and†

$$\begin{aligned} m_j &= \tfrac{1}{2}[\mu_{j+\frac{1}{2}} l_{j+\frac{1}{2}} + \mu_{j-\frac{1}{2}} l_{j-\frac{1}{2}}] \\ e_{j+\frac{1}{2}} &= \rho k_{j+\frac{1}{2}} l_{j+\frac{1}{2}} \sigma_{j+\frac{1}{2}} \\ m_j^X &= m_j^* + \rho V_j^X \\ m_j^Y &= m_j^* + \rho V_j^Y \\ \cos \theta_{j+\frac{1}{2}} &= (x_{j+1} - x_j)/l_{j+\frac{1}{2}} \\ \sin \theta_{j+\frac{1}{2}} &= (y_{j+1} - y_j)/l_{j+\frac{1}{2}} \,. \end{aligned}$$

Each lumped mass, m_j, has been expressed as the average mass of the two segments of cable which lie on either side of station j. Also, one-half the equivalent transverse mass, e_j, of the fluid entrained with each of these segments has been included in the inertia tensor. Furthermore, at those stations from which a weight is suspended, the effective horizontal and vertical masses, m_j^X and m_j^Y, of the weight are to be added. For simplicity in allowing for virtual inertia, we have assumed that any such weights possess a certain degree of symmetry and remain upright as the line moves about.

The force vector, F_j, on the right side of eq. (2.1) can be expressed as the sum of internal forces (the tensions acting between adjacent mass elements) and whatever external forces are present. Thus, in expanded form the equations of motion can be written

$$\begin{aligned} I_j\ddot{x}_j - K_j\ddot{y}_j &= T_{j+\frac{1}{2}} \cos \theta_{j+\frac{1}{2}} - T_{j-\frac{1}{2}} \cos \theta_{j-\frac{1}{2}} + X_j \\ -K_j\ddot{x}_j + J_j\ddot{y}_j &= T_{j+\frac{1}{2}} \sin \theta_{j+\frac{1}{2}} - T_{j-\frac{1}{2}} \sin \theta_{j-\frac{1}{2}} + Y_j \end{aligned} \tag{2.2}$$

where $T_{j+\frac{1}{2}}$ = tension in segment of line between stations j and $j + 1$
X_j = horizontal component of resultant external force at station j
Y_j = vertical component of resultant external force at station j.

There are two sources of external force, namely: 1) gravity, which gives rise to the weight minus the buoyancy and acts only in the vertical, and 2) fluid resistance, which gives rise to the damping forces. Thus, we write

$$\begin{aligned} X_j &= -\tfrac{1}{2}[D_{j+\frac{1}{2}} \sin \theta_{j+\frac{1}{2}} + D_{j-\frac{1}{2}} \sin \theta_{j-\frac{1}{2}}] + X_j^* \\ Y_j &= \tfrac{1}{2}[D_{j+\frac{1}{2}} \cos \theta_{j+\frac{1}{2}} + D_{j-\frac{1}{2}} \cos \theta_{j-\frac{1}{2}}] + Y_j^* - W_j - W_j^* \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} W_j &= m_j g - \tfrac{1}{2}\rho g(l_{j+\frac{1}{2}}\sigma_{j+\frac{1}{2}} + l_{j-\frac{1}{2}}\sigma_{j-\frac{1}{2}}) \\ W_j^* &= m_j^* g - \rho g V_j^* \end{aligned}$$

† See Appendix for definitions of the parameters which appear in the formulas.

and $D_{j+\frac{1}{2}}$ = drag on segment of line between stations j and $j + 1$
X_j^* = horizontal component of damping force on weight at station j
Y_j^* = vertical component of damping force on weight at station j.

Again, in order to get the best approximation to the continuous case, the net effect of the drag at station j has been expressed as one-half of each component of the drag on the segments which lie on either side of this station. The buoyant force of the displaced fluid has been treated likewise.

We have assumed that the drag, $D_{j+\frac{1}{2}}$, on a segment of the line acts in a direction at right angles to the line. This is a good approximation whenever the velocity is high enough to produce significant forces, since at all but the lowest Reynolds numbers the tangential component of the hydrodynamic force is very small compared to the normal component. Furthermore, we assume that the drag is proportional to the *square* of the component of relative velocity *normal* to the line:

$$D_{j+\frac{1}{2}} = -f^D_{j+\frac{1}{2}} q_{j+\frac{1}{2}} \mid q_{j+\frac{1}{2}} \mid \tag{2.4}$$

where

$$f^D_{j+\frac{1}{2}} = \tfrac{1}{2}\rho C^D_{j+\frac{1}{2}} l_{j+\frac{1}{2}} d_{j+\frac{1}{2}}$$

$$q_{j+\frac{1}{2}} = -\tfrac{1}{2}[(\dot{x}_{j+1} - c) + (\dot{x}_j - c)] \sin \theta_{j+\frac{1}{2}} + \tfrac{1}{2}[\dot{y}_{j+1} + \dot{y}_j] \cos \theta_{j+\frac{1}{2}} .$$

The positive normal to the line has been arbitrarily taken to be directed upward when θ equals zero. The use of the minus sign and the introduction of the absolute value of one of the velocity factors ensures that the drag will always be opposed to the direction of $q_{j+\frac{1}{2}}$ and thus act as a dissipative force to remove energy from the system. Since the velocities of the two endpoints of each segment will, in general, differ slightly, their mean value (which for a straight line segment is exactly equal to the velocity of the midpoint) is taken as a representative value in the definition of $q_{j+\frac{1}{2}}$. In addition, the definition allows for the presence of a uniform horizontal current, c, to incorporate the ability to treat towing lines as well as mooring lines (or mooring lines subjected to ocean currents).

In addition to the drag on the line itself, there will also be resistance to the motion of any concentrated loads which may be suspended from the line. These additional damping forces will vary with the velocity but will not, in general, be directed exactly opposite to the motion of each weight. However, on account of the assumed orientation and symmetry of any such weights, the resistance force will be parallel to the velocity vector whenever the relative motion is either purely horizontal or purely vertical. Accordingly, the two components of resistance may be written

$$\begin{aligned} X_j^* &= -f_j^X u_j(\dot{x}_j - c) \\ Y_j^* &= -f_j^Y u_j \dot{y}_j \end{aligned} \tag{2.5}$$

where

$$f_j^X = \tfrac{1}{2}\rho C_j^X S_j^X$$

$$f_j^Y = \tfrac{1}{2}\rho C_j^Y S_j^Y$$

$$u_j = [(\dot{x}_j - c)^2 + \dot{y}_j^2]^{\frac{1}{2}}$$

Up to this point an explicit formula has been given for the evaluation of every term in the equations of motion (2.2) with the exception of the tensions. To determine these we must invoke the inextensibility condition which was assumed at the outset. This takes the form of a constraint on the motion of the line. It requires that the separation between adjacent stations must not change with time. Thus, we write

$$(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 = l_{j-\frac{1}{2}}^2 = \text{const.} \tag{2.6}$$

This holds for each segment of the line, and we require that the corresponding set of tensions, $T_{j-\frac{1}{2}}$, take on values such that the resulting solution of the equations of motion will be consistent with eq. (2.6). Because of the implicit nature of this condition, we are led to a system of algebraic equations for the determination of the proper tensions. At the extremities of the line ($j = 0$ and $j = s$) x_j and y_j must be obtained from the boundary conditions, namely:

$$\begin{aligned} x_0 &= x_0(t), \qquad & y_0 &= y_0(t) \\ x_s &= x_s(t), \qquad & y_s &= y_s(t). \end{aligned} \tag{2.7}$$

These are given as functions of time, and permit the introduction of any desired types of driving motions.

Finally, to complete the formulation of the problem a set of initial conditions must be given for each station on the line. Since the equations of motion are of the second order, it is necessary to specify both the coordinates and the velocities at $t = 0$. That is,

$$\begin{aligned} x_j(0) &= x_j^0, \qquad & y_j(0) &= y_j^0 \qquad & (j = 1, 2, \cdots s - 1) \\ \dot{x}_j(0) &= \dot{x}_j^0, \qquad & \dot{y}_j(0) &= \dot{y}_j^0 \qquad & (j = 1, 2, \cdots s - 1) \end{aligned} \tag{2.8}$$

where the superscript index "0" is used to designate a value at the origin in time.

3. Solution of Equations by Finite Differences.

A. General Description of Computational Procedure. The equations governing the motion of a cable, as derived in the last section, are summarized here. The basic equations of motion, (2.2), are repeated for convenience,

$$\begin{aligned} I_j\ddot{x}_j - K_j\ddot{y}_j &= T_{j+\frac{1}{2}} \cos \theta_{j+\frac{1}{2}} - T_{j-\frac{1}{2}} \cos \theta_{j-\frac{1}{2}} + X_j \\ &\qquad\qquad (j = 1, 2, \cdots s - 1), \\ -K_j\ddot{x}_j + J_j\ddot{y}_j &= T_{j+\frac{1}{2}} \sin \theta_{j+\frac{1}{2}} - T_{j-\frac{1}{2}} \sin \theta_{j-\frac{1}{2}} + Y_j \\ &\qquad\qquad (j = 1, 2, \cdots s - 1), \end{aligned} \tag{2.2}$$

where

a) s is the number of segments into which the cable is divided,

b) I_j, K_j, J_j are given in eq. (2.1) and are functions of the physical properties of the cable and of position only,

c) X_j, Y_j are given by eqs. (2.3), (2.4), and (2.5) and are functions of the physical properties of the cable and of position and velocity.

In addition, the motion is governed by the condition of inextensibility of the cable,

eq. (2.6), namely,

$$(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 = l_{j-\frac{1}{2}}^2 = \text{const.} \qquad (j = 1, 2, \cdots s). \tag{2.6}$$

The differentiated (with respect to time) forms of this relation,

$$(x_j - x_{j-1})(\dot{x}_j - \dot{x}_{j-1}) + (y_j - y_{j-1})(\dot{y}_j - \dot{y}_{j-1}) = 0 \qquad (j = 1, 2, \cdots s), \tag{3.1}$$

$$\begin{aligned}(x_j - x_{j-1})(\ddot{x}_j - \ddot{x}_{j-1}) + (y_j - y_{j-1})(\ddot{y}_j - \ddot{y}_{j-1})\\ + (\dot{x}_j - \dot{x}_{j-1})^2 + (\dot{y}_j - \dot{y}_{j-1})^2 = 0 \qquad (j = 1, 2, \cdots s),\end{aligned} \tag{3.2}$$

are also used in the computation.

For numerical solution by finite-difference methods the following finite-difference equivalents are used,

$$\dot{x}_j^{n+\frac{1}{2}} = \frac{x_j^{n+1} - x_j^n}{\Delta t}, \qquad \dot{y}_j^{n+\frac{1}{2}} = \frac{y_j^{n+1} - y_j^n}{\Delta t} \qquad (j = 1, 2, \cdots s - 1), \tag{3.3}$$

$$\ddot{x}_j^n = \frac{x_j^{n+1} - 2x_j^n + x_j^{n-1}}{(\Delta t)^2}, \qquad \ddot{y}_j^n = \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{(\Delta t)^2} \qquad (j = 1, 2, \cdots s - 1). \tag{3.4}$$

It is assumed that the boundary and initial conditions are known. These are given in eqs. (2.7) and (2.8), respectively. The system of equations summarized above, consisting of eqs. (2.2), (2.6), (2.7), (2.8), (3.1), (3.2), (3.3), (3.4) with the auxiliary equations (2.1), (2.3), (2.4), (2.5), completely describe the motion of the cable.

The computational procedure, as developed in detail in the remainder of this section, consists of an algorithm to determine the values x_j^{n+1}, y_j^{n+1}, $\dot{x}_j^{n+\frac{1}{2}}$, $\dot{y}_j^{n+\frac{1}{2}}$ (at time $t = t^{n+1} = t^n + \Delta t$ and $t^{n+\frac{1}{2}} = t^n + \frac{1}{2}\Delta t$) from known values x_j^n, y_j^n, $\dot{x}_j^{n-\frac{1}{2}}$, $\dot{y}_j^{n-\frac{1}{2}}$ (at time $t = t^n$ and $t^{n-\frac{1}{2}}$). It is convenient to divide this algorithm in two phases. The first phase involves the determination of a tentative (but consistent) set of tensions for all segments, and numerical integration of the equations of motion to predict the coordinates one step ahead; the second phase involves the evaluation of small discrepancies in the constraint equations from which a set of first-order corrections to the tensions can be obtained, and integration of the equations a second time to obtain accurate values of the coordinates.

Phase 1. Using eqs. (2.2) and (3.2), ($3s - 2$ equations), we compute the ($3s - 2$) unknown variables $\ddot{x}_j^n$, $\ddot{y}_j^n$ ($j = 1, 2, \cdots s - 1$) and $T_{j-\frac{1}{2}}^n$ ($j = 1, 2, \cdots s$). We now use eqs. (3.3) and (3.4), ($4s - 4$ equations), to compute the ($4s - 4$) variables at the next time step, x_j^{n+1}, y_j^{n+1}, $\dot{x}_j^{n+\frac{1}{2}}$, $\dot{y}_j^{n+\frac{1}{2}}$ ($j = 1, 2 \cdots s - 1$). These are considered to be only tentative values (denoted in subsequent text by use of the tilde).

Phase 2. To obtain improved values of the tensions $T_{j-\frac{1}{2}}^n$ ($j = 1, 2, \cdots s$) and the quantities $\ddot{x}_j^n$, $\ddot{y}_j^n$, x_j^{n+1}, y_j^{n+1}, $\dot{x}_j^{n+\frac{1}{2}}$, $\dot{y}_j^{n+\frac{1}{2}}$ (a total of ($7s - 6$) quantities) we use the system of equations (2.2), (2.6) and (3.3), (3.4), consisting of ($7s - 6$) equations. However, since eqs. (2.6) are not linear but quadratic in the unknowns x_j and y_j, an explicit solution is impractical to obtain. For this reason a computation algorithm based on the Newton-Raphson method of successive approximations

is developed. A detailed discussion of the computational procedure used in this problem is given in the sections which follow.

B. Determination of Tentative Values of Tensions. The system of equations (2.2) may be regarded as a set of $(2s - 2)$ linear equations in the variables $\ddot{x}_j$ and $\ddot{y}_j$ (accelerations) and may be solved directly for these variables. If we designate

$$L_j = (\Delta t)^2 I_j/(I_j J_j - K_j^2)$$
$$M_j = (\Delta t)^2 J_j/(I_j J_j - K_j^2)$$
$$N_j = (\Delta t)^2 K_j/(I_j J_j - K_j^2),$$

then the equations of motion (2.2) can be reduced to:

$$\begin{aligned} \ddot{x}_j &= [R_j T_{j+\frac{1}{2}} - P_j T_{j-\frac{1}{2}} + U_j]/(\Delta t)^2 \\ \ddot{y}_j &= [S_j T_{j+\frac{1}{2}} - Q_j T_{j-\frac{1}{2}} + V_j]/(\Delta t)^2 \end{aligned} \tag{3.5}$$

where

$$P_j = M_j \cos \theta_{j-\frac{1}{2}} + N_j \sin \theta_{j-\frac{1}{2}}$$
$$Q_j = N_j \cos \theta_{j-\frac{1}{2}} + L_j \sin \theta_{j-\frac{1}{2}}$$
$$R_j = M_j \cos \theta_{j+\frac{1}{2}} + N_j \sin \theta_{j+\frac{1}{2}}$$
$$S_j = N_j \cos \theta_{j+\frac{1}{2}} + L_j \sin \theta_{j+\frac{1}{2}}$$
$$U_j = M_j X_j + N_j Y_j$$
$$V_j = N_j X_j + L_j Y_j .$$

We observe that eq. (3.2) involves positions, velocities, and accelerations. As is often the case with finite-difference procedures, it proves to be convenient to compute positions and accelerations at the mesh points while velocities are found at the midpoints in time. For this reason we shall use a modified form obtained by evaluating eq. (3.2) at $t = t^n$ and at $t = t^{n-1}$, and then adding the two results together, namely,

$$\begin{aligned} &(x_j^n - x_{j-1}^n)(\ddot{x}_j^n - \ddot{x}_{j-1}^n) + (y_j^n - y_{j-1}^n)(\ddot{y}_j^n - \ddot{y}_{j-1}^n) \\ &+ (x_j^{n-1} - x_{j-1}^{n-1})(\ddot{x}_j^{n-1} - \ddot{x}_{j-1}^{n-1}) + (y_j^{n-1} - y_{j-1}^{n-1})(\ddot{y}_j^{n-1} - \ddot{y}_{j-1}^{n-1}) \\ &+ 2(\Delta t)^{-2}[(x_j^n - x_j^{n-1}) - (x_{j-1}^n - x_{j-1}^{n-1})]^2 \\ &+ 2(\Delta t)^{-2}[(y_j^n - y_j^{n-1}) - (y_{j-1}^n - y_{j-1}^{n-1})]^2 = 0, \end{aligned} \tag{3.6}$$

in which we have used the approximations,

$$\begin{aligned} (\dot{x}_j^n - \dot{x}_{j-1}^n)^2 + (\dot{x}_j^{n-1} - \dot{x}_{j-1}^{n-1})^2 &\approx 2(\dot{x}_j^{n-\frac{1}{2}} - \dot{x}_{j-1}^{n-\frac{1}{2}})^2 \\ &\approx 2[(x_j^n - x_j^{n-1})/\Delta t - (x_{j-1}^n - x_{j-1}^{n-1})/\Delta t]^2, \end{aligned}$$

$$\begin{aligned} (\dot{y}_j^n - \dot{y}_{j-1}^n)^2 + (\dot{y}_j^{n-1} - \dot{y}_{j-1}^{n-1})^2 &\approx 2(\dot{y}_j^{n-\frac{1}{2}} - \dot{y}_{j-1}^{n-\frac{1}{2}})^2 \\ &\approx 2[(y_j^n - y_j^{n-1})/\Delta t - (y_{j-1}^n - y_{j-1}^{n-1})/\Delta t]^2. \end{aligned}$$

Note that eqs. (3.6) are linear in the accelerations. Likewise, eqs. (3.5) are linear in the tensions. Consequently, when these latter expressions are substituted for the

acceleration components in the constraint equations (3.6), we obtain a set of conditions which are linear in the tensions, namely,

$$
\begin{aligned}
(3.7)\qquad & E^n_{j-\frac{1}{2}}\,\tilde{T}^n_{j-\frac{3}{2}} - F^n_{j-\frac{1}{2}}\,\tilde{T}^n_{j-\frac{1}{2}} + G^n_{j-\frac{1}{2}}\,\tilde{T}^n_{j+\frac{1}{2}} \\
& + E^{n-1}_{j-\frac{1}{2}}\,T^{n-1}_{j-\frac{3}{2}} - F^{n-1}_{j-\frac{1}{2}}\,T^{n-1}_{j-\frac{1}{2}} + G^{n-1}_{j-\frac{1}{2}}\,T^{n-1}_{j+\frac{1}{2}} + H^n_{j-\frac{1}{2}} + H^{n-1}_{j-\frac{1}{2}} \\
& + 2[(x_j^n - x_j^{n-1}) - (x^n_{j-1} - x^{n-1}_{j-1})]^2 \\
& \qquad\qquad + 2[(y_j^n - y_j^{n-1}) - (y^n_{j-1} - y^{n-1}_{j-1})]^2 = 0
\end{aligned}
$$

where

$$
\begin{aligned}
E^n_{j-\frac{1}{2}} &= (x_j^n - x^n_{j-1})P^n_{j-1} + (y_j^n - y^n_{j-1})Q^n_{j-1} \\
F^n_{j-\frac{1}{2}} &= (x_j^n - x^n_{j-1})\,(P_j^n + R^n_{j-1}) + (y_j^n - y^n_{j-1})\,(Q_j^n + S^n_{j-1}) \\
G^n_{j-\frac{1}{2}} &= (x_j^n - x^n_{j-1})\,R_j^n + (y_j^n - y^n_{j-1})\,S_j^n \\
H^n_{j-\frac{1}{2}} &= (x_j^n - x^n_{j-1})\,(U_j^n - U^n_{j-1}) + (y_j^n - y^n_{j-1})\,(V_j^n - V^n_{j-1}).
\end{aligned}
$$

Now assume that the solution is correct up to $t = t^n$. Then all quantities in (3.7) can be evaluated at once except for $\tilde{T}^n_{j-\frac{3}{2}}$, $\tilde{T}^n_{j-\frac{1}{2}}$ and $\tilde{T}^n_{j+\frac{1}{2}}$. The tentative values of the tensions—signified by the tildes—are determined by the following system of equations:

$$
(3.8)\qquad
\begin{bmatrix}
-F^n_{0.5} & G^n_{0.5} & & & & \\
E^n_{1.5} & -F^n_{1.5} & G^n_{1.5} & & & \\
 & E^n_{2.5} & -F^n_{2.5} & G^n_{2.5} & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & E^n_{s-1.5} & -F^n_{s-1.5} & G^n_{s-1.5} \\
 & & & & E^n_{s-0.5} & -F^n_{s-0.5}
\end{bmatrix}
\begin{bmatrix}
\tilde{T}^n_{0.5} \\ \tilde{T}^n_{1.5} \\ \tilde{T}^n_{2.5} \\ \vdots \\ \tilde{T}^n_{s-1.5} \\ \tilde{T}^n_{s-0.5}
\end{bmatrix}
=
\begin{bmatrix}
-\Psi^n_{0.5} \\ -\Psi^n_{1.5} \\ -\Psi^n_{2.5} \\ \vdots \\ -\Psi^n_{s-1.5} \\ -\Psi^n_{s-0.5}
\end{bmatrix}
$$

where

$$
\begin{aligned}
\Psi^n_{j-\frac{1}{2}} = {} & E^{n-1}_{j-\frac{1}{2}}\,T^{n-1}_{j-\frac{3}{2}} - F^{n-1}_{j-\frac{1}{2}}\,T^{n-1}_{j-\frac{1}{2}} + G^{n-1}_{j-\frac{1}{2}}\,T^{n-1}_{j+\frac{1}{2}} + H^{n-1}_{j-\frac{1}{2}} + H^n_{j-\frac{1}{2}} \\
& + 2[(x_j^n - x_j^{n-1}) - (x^n_{j-1} - x^{n-1}_{j-1})]^2 + 2[(y_j^n - y_j^{n-1}) - (y^n_{j-1} - y^{n-1}_{j-1})]^2.
\end{aligned}
$$

In general, we can write (for $j = 1, 2, \cdots s$)

$$
(3.9)\qquad E^n_{j-\frac{1}{2}}\,\tilde{T}^n_{j-\frac{3}{2}} - F^n_{j-\frac{1}{2}}\,\tilde{T}^n_{j-\frac{1}{2}} + G^n_{j-\frac{1}{2}}\,\tilde{T}^n_{j+\frac{1}{2}} + \Psi^n_{j-\frac{1}{2}} = 0
$$

with the conditions: $E^n_{0.5} = G^n_{s-\frac{1}{2}} = 0$ for all n. Also, P_0^n, Q_0^n, R_0^n, S_0^n, and P_s^n, Q_s^n, R_s^n, $S_s^n = 0$ for all n; and

$$
\begin{aligned}
U_0^n &= (\Delta t)^2\ddot{x}_0^n \quad\text{and}\quad U_s^n = (\Delta t)^2\ddot{x}_s^n \quad\text{for all } n, \\
V_0^n &= (\Delta t)^2\ddot{y}_0^n \quad\text{and}\quad V_s^n = (\Delta t)^2\ddot{y}_s^n \quad\text{for all } n.
\end{aligned}
$$

The matrix of coefficients of the system of equations is a triple diagonal one, and it can be easily reduced to a single linear equation by elimination. Thus, we solve eq. (3.9) for $\tilde{T}^n_{j+\frac{1}{2}}$.

$$
(3.10)\qquad \tilde{T}^n_{j+\frac{1}{2}} = \frac{F^n_{j-\frac{1}{2}}}{G^n_{j-\frac{1}{2}}}\,\tilde{T}^n_{j-\frac{1}{2}} - \frac{E^n_{j-\frac{1}{2}}}{G^n_{j-\frac{1}{2}}}\,\tilde{T}^n_{j-\frac{3}{2}} - \frac{\Psi^n_{j-\frac{1}{2}}}{G^n_{j-\frac{1}{2}}}
$$

Now we express each tension as a linear function of $\tilde{T}^n_{0.5}$ (the tension in the first segment) as follows:

$$\tilde{T}^n_{j+\frac{1}{2}} = \alpha^n_{j+\frac{1}{2}} \tilde{T}^n_{0.5} + \beta^n_{j+\frac{1}{2}} \tag{3.11}$$

and we arrive at the following recursion formulas for $\alpha^n_{j+\frac{1}{2}}$ and $\beta^n_{j+\frac{1}{2}}$, namely,

$$\begin{aligned} \alpha^n_{j+\frac{1}{2}} &= (F^n_{j-\frac{1}{2}} \alpha^n_{j-\frac{1}{2}} - E^n_{j-\frac{1}{2}} \alpha^n_{j-\frac{3}{2}})/G^n_{j-\frac{1}{2}} \\ \beta^n_{j+\frac{1}{2}} &= (F^n_{j-\frac{1}{2}} \beta^n_{j-\frac{1}{2}} - E^n_{j-\frac{1}{2}} \beta^n_{j-\frac{3}{2}} - \Psi^n_{j-\frac{1}{2}})/G^n_{j-\frac{1}{2}} \end{aligned} \tag{3.12}$$

with the conditions

$$\alpha^n_{0.5} = 1, \qquad \alpha^n_{-0.5} = 0 \quad \text{for all } n,$$

$$\beta^n_{0.5} = 0, \qquad \beta^n_{-0.5} = 0 \quad \text{for all } n.$$

Starting with $j = 1$, we evaluate $\alpha^n_{j+\frac{1}{2}}$ and $\beta^n_{j+\frac{1}{2}}$ recursively up to $j = s - 1$. We then find $\tilde{T}^n_{0.5}$ from the last equation of the system.

$$\tilde{T}^n_{0.5} = -\frac{(F^n_{s-\frac{1}{2}} \beta^n_{s-\frac{1}{2}} - E^n_{s-\frac{1}{2}} \beta^n_{s-\frac{3}{2}} - \Psi^n_{s-\frac{1}{2}})}{(F^n_{s-\frac{1}{2}} \alpha^n_{s-\frac{1}{2}} - E^n_{s-\frac{1}{2}} \alpha^n_{s-\frac{3}{2}})}. \tag{3.13}$$

C. Computation of Tentative Coordinates. In order to solve eqs. (3.5) numerically, we replace $\ddot{x}_j^n$ and $\ddot{y}_j^n$ by their simplest central-difference approximations, eq. (3.4), namely,

$$\begin{aligned} \ddot{x}_j^n &= (x_j^{n+1} - 2x_j^n + x_j^{n-1})/(\Delta t)^2 \\ \ddot{y}_j^n &= (y_j^{n+1} - 2y_j^n + y_j^{n-1})/(\Delta t)^2. \end{aligned} \tag{3.14}$$

Now we solve for x_j^{n+1} and y_j^{n+1}, considering these as tentative values subject to a slight modification in order to satisfy a system of constraints. Thus we write

$$\begin{aligned} \tilde{x}_j^{n+1} &= 2x_j^n - x_j^{n-1} - P_j^n \tilde{T}^n_{j-\frac{1}{2}} + R_j^n \tilde{T}^n_{j+\frac{1}{2}} + U_j^n \\ \tilde{y}_j^{n+1} &= 2y_j^n - y_j^{n-1} - Q_j^n \tilde{T}^n_{j-\frac{1}{2}} + S_j^n \tilde{T}^n_{j+\frac{1}{2}} + V_j^n. \end{aligned} \tag{3.15}$$

The quantities P_j^n, Q_j^n, R_j^n, S_j^n, U_j^n and V_j^n are the same as were used to set up the coefficient matrix for the tensions, and the values for $\tilde{T}^n_{j-\frac{1}{2}}$ and $\tilde{T}^n_{j+\frac{1}{2}}$ are obtained from (3.11).

D. Determination of Improved Values of Tensions. Next, we determine the set of corrections $\delta T^n_{j-\frac{1}{2}}$ to be applied to the tensions $\tilde{T}^n_{j-\frac{1}{2}}$ in order that the values of x_j^{n+1} and y_j^{n+1} should also satisfy the inextensibility condition (2.6). For this purpose we define the function

$$\Omega^{n+1}_{j-\frac{1}{2}} \equiv \tfrac{1}{2}[(x_j^{n+1} - x_{j-1}^{n+1})^2 + (y_j^{n+1} - y_{j-1}^{n+1})^2 - l^2_{j-\frac{1}{2}}] \tag{3.16}$$

which measures the discrepancy in the distance between the extrapolated positions of pairs of adjacent stations. We observe from eqs. (3.15)—with the tildes suppressed—that x_j^{n+1} and y_j^{n+1} are functions of the tensions. Consequently, $\Omega^{n+1}_{j-\frac{1}{2}}$ may also be expressed as a function of the tensions. This enables us to write the system of constraints to which the tensions are subject as follows:

$$\Omega^{n+1}_{j-\frac{1}{2}} = \Omega^{n+1}_{j-\frac{1}{2}} \{T^n_{j-\frac{3}{2}}, T^n_{j-\frac{1}{2}}, T^n_{j+\frac{1}{2}}\} = 0 \qquad (j = 1, 2, \cdots s) \tag{3.17}$$

since $\Omega^{n+1}_{j-\frac{1}{2}}$ vanishes when the inextensibility condition is obeyed.

Now let

$$T^n_{j+\frac12} = \tilde T^n_{j+\frac12} + \delta T^n_{j+\frac12}, \tag{3.18}$$

and expand $\Omega^{n+1}_{j-\frac12}$ in a Taylor series about the point $\{\tilde T^n_{j-\frac32}, \tilde T^n_{j-\frac12}, \tilde T^n_{j+\frac12}\}$. Thus, we obtain

$$\begin{aligned}\Omega^{n+1}_{j-\frac12} = \tilde\Omega^{n+1}_{j-\frac12} + \frac{\partial\tilde\Omega^{n+1}_{j-\frac12}}{\partial T^n_{j-\frac32}}\,\delta T^n_{j-\frac32} + \frac{\partial\tilde\Omega^{n+1}_{j-\frac12}}{\partial T^n_{j-\frac12}}\,\delta T^n_{j-\frac12} \\ + \frac{\partial\tilde\Omega^{n+1}_{j-\frac12}}{\partial T^n_{j+\frac12}}\,\delta T^n_{j+\frac12} + \text{higher order terms,}\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\tilde\Omega^{n+1}_{j-\frac12} &\equiv \Omega^{n+1}_{j-\frac12}\{\tilde T^n_{j-\frac32}, \tilde T^n_{j-\frac12}, \tilde T^n_{j+\frac12}\} \\ &= \tfrac12[(\tilde x^{n+1}_j - \tilde x^{n+1}_{j-1})^2 + (\tilde y^{n+1}_j - \tilde y^{n+1}_{j-1})^2 - l^2_{j-\frac12}].\end{aligned}$$

Provided that the tentative values $\tilde T^n_{j-\frac12}$ are sufficiently close to the correct values $T^n_{j-\frac12}$, we may neglect the higher order terms in the expansion (3.19), and thereby obtain a system of s linear equations for the differential corrections $\delta T^n_{j-\frac12}$. These equations have the same form as the previous system (3.8) for determining $\tilde T^n_{j-\frac12}$, namely,

$$\begin{bmatrix} -\tilde F^{n+1}_{0.5} & \tilde G^{n+1}_{0.5} & & & & \\ \tilde E^{n+1}_{1.5} & -\tilde F^{n+1}_{1.5} & \tilde G^{n+1}_{1.5} & & & \\ & \tilde E^{n+1}_{2.5} & -\tilde F^{n+1}_{2.5} & \tilde G^{n+1}_{2.5} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tilde E^{n+1}_{s-1.5} & -\tilde F^{n+1}_{s-1.5} & \tilde G^{n+1}_{s-1.5} \\ & & & & \tilde E^{n+1}_{s-0.5} & -\tilde F^{n+1}_{s-0.5} \end{bmatrix} \begin{bmatrix} \delta T^n_{0.5} \\ \delta T^n_{1.5} \\ \delta T^n_{2.5} \\ \vdots \\ \delta T^n_{s-1.5} \\ \delta T^n_{s-0.5} \end{bmatrix} = \begin{bmatrix} -\tilde\Omega^{n+1}_{0.5} \\ -\tilde\Omega^{n+1}_{1.5} \\ -\tilde\Omega^{n+1}_{2.5} \\ \vdots \\ -\tilde\Omega^{n+1}_{s-1.5} \\ -\tilde\Omega^{n+1}_{s-0.5} \end{bmatrix} \tag{3.20}$$

the general expression being (for $j = 1, 2, \cdots s$)

$$\tilde E^{n+1}_{j-\frac12}\,\delta T^n_{j-\frac32} - \tilde F^{n+1}_{j-\frac12}\,\delta T^n_{j-\frac12} + \tilde G^{n+1}_{j-\frac12}\,\delta T^n_{j+\frac12} + \tilde\Omega^{n+1}_{j-\frac12} = 0 \tag{3.21}$$

where

$$\begin{aligned}\tilde E^{n+1}_{j-\frac12} &= \frac{\partial\tilde\Omega^{n+1}_{j-\frac12}}{\partial T^n_{j-\frac32}} = (\tilde x^{n+1}_j - \tilde x^{n+1}_{j-1})P^n_{j-1} + (\tilde y^{n+1}_j - \tilde y^{n+1}_{j-1})Q^n_{j-1} \\ \tilde F^{n+1}_{j-\frac12} &= -\frac{\partial\tilde\Omega^{n+1}_{j-\frac12}}{\partial T^n_{j-\frac12}} = (\tilde x^{n+1}_j - \tilde x^{n+1}_{j-1})(P^n_j + R^n_{j-1}) + (\tilde y^{n+1}_j - \tilde y^{n+1}_{j-1})(Q^n_j + S^n_{j-1}) \\ \tilde G^{n+1}_{j-\frac12} &= \frac{\partial\tilde\Omega^{n+1}_{j-\frac12}}{\partial T^n_{j+\frac12}} = (\tilde x^{n+1}_j - \tilde x^{n+1}_{j-1})R^n_j + (\tilde y^{n+1}_j - \tilde y^{n+1}_{j-1})S^n_j\end{aligned}$$

with the conditions: $\tilde E^{n+1}_{0.5} = \tilde G^{n+1}_{s-\frac12} = 0$ for all n, and the quantities P_j^n, Q_j^n, R_j^n and S_j^n being the same as in (3.7).

The system (3.20) can be solved in a manner completely analogous to the solution of the system (3.8). Thus, we write

$$\delta T^n_{j+\frac12} = \kappa^n_{j+\frac12}\,\delta T^n_{0.5} + \lambda^n_{j+\frac12} \tag{3.22}$$

and obtain the following recursion formulas:

$$\begin{aligned} \kappa_{j+\frac{1}{2}}^{n} &= (\tilde{F}_{j-\frac{1}{2}}^{n+1}\kappa_{j-\frac{1}{2}}^{n} - \tilde{E}_{j-\frac{1}{2}}^{n+1}\kappa_{j-\frac{1}{2}}^{n})/\tilde{G}_{j-\frac{1}{2}}^{n+1} \\ \lambda_{j+\frac{1}{2}}^{n} &= (\tilde{F}_{j-\frac{1}{2}}^{n+1}\lambda_{j-\frac{1}{2}}^{n} - \tilde{E}_{j-\frac{1}{2}}^{n+1}\lambda_{j-\frac{1}{2}}^{n} - \tilde{\Omega}_{j-\frac{1}{2}}^{n+1})/\tilde{G}_{j-\frac{1}{2}}^{n+1} \end{aligned} \tag{3.23}$$

with the conditions

$$\kappa_{0.5}^{n} = 1, \qquad \kappa_{-0.5}^{n} = 0 \quad \text{for all } n,$$

$$\lambda_{0.5}^{n} = 0, \qquad \lambda_{-0.5}^{n} = 0 \quad \text{for all } n.$$

Finally, the last equation of the system enables us to solve for $\delta T_{0.5}^{n}$. The result is

$$\delta T_{0.5}^{n} = -\frac{(\tilde{F}_{s-\frac{1}{2}}^{n+1}\lambda_{s-\frac{1}{2}}^{n} - \tilde{E}_{s-\frac{1}{2}}^{n+1}\lambda_{s-\frac{1}{2}}^{n} - \tilde{\Omega}_{s-\frac{1}{2}}^{n+1})}{(\tilde{F}_{s-\frac{1}{2}}^{n+1}\kappa_{s-\frac{1}{2}}^{n} - \tilde{E}_{s-\frac{1}{2}}^{n+1}\kappa_{s-\frac{1}{2}}^{n})}. \tag{3.24}$$

We can now obtain the corrected values of the tension in each segment. Thus,

$$\begin{aligned} T_{j-\frac{1}{2}}^{n} &= \tilde{T}_{j-\frac{1}{2}}^{n} + \delta T_{j-\frac{1}{2}}^{n} \\ &= \tilde{T}_{j-\frac{1}{2}}^{n} + \kappa_{j-\frac{1}{2}}^{n}\delta T_{0.5}^{n} + \lambda_{j-\frac{1}{2}}^{n}. \end{aligned} \tag{3.25}$$

E. Computation of Improved Coordinates. The corrected values of the coordinates are found using eqs. (3.15)—but this time with the tildes suppressed—namely:

$$\begin{aligned} x_j^{n+1} &= 2x_j^{n} - x_j^{n-1} - P_j^{n}T_{j-\frac{1}{2}}^{n} + R_j^{n}T_{j+\frac{1}{2}}^{n} + U_j^{n} \\ y_j^{n+1} &= 2y_j^{n} - y_j^{n-1} - Q_j^{n}T_{j-\frac{1}{2}}^{n} + S_j^{n}T_{j+\frac{1}{2}}^{n} + V_j^{n}. \end{aligned} \tag{3.26}$$

For solution on an automatic computer, it is more convenient to express eqs. (3.26) in terms of corrections to be added to the tentative values of the coordinates. That is,

$$\begin{aligned} \delta x_j^{n+1} &= -P_j^{n}\delta T_{j-\frac{1}{2}}^{n} + R_j^{n}\delta T_{j+\frac{1}{2}}^{n} \\ \delta y_j^{n+1} &= -Q_j^{n}\delta T_{j-\frac{1}{2}}^{n} + S_j^{n}\delta T_{j+\frac{1}{2}}^{n}. \end{aligned} \tag{3.27}$$

Then the corrected coordinates are given by:

$$\begin{aligned} x_j^{n+1} &= \tilde{x}_j^{n+1} + \delta x_j^{n+1} \\ y_j^{n+1} &= \tilde{y}_j^{n+1} + \delta y_j^{n+1}. \end{aligned} \tag{3.28}$$

F. Special Form of Equations for Computing First Time Step. We assume that the initial velocity components are zero at each station, and we obtain the initial coordinates from the equations for *static* equilibrium of the line. Since $\dot{x}_j^0$ and $\dot{y}_j^0 = 0$, eq. (3.2) reduces to

$$(x_j^0 - x_{j-1}^0)(\ddot{x}_j^0 - \ddot{x}_{j-1}^0) + (y_j^0 - y_{j-1}^0)(\ddot{y}_j^0 - \ddot{y}_{j-1}^0) = 0 \tag{3.29}$$

and, on substituting the expressions (3.5), we find that the tensions are subject to the constraint

$$E_{j-\frac{1}{2}}^{0}\,\tilde{T}_{j-\frac{1}{2}}^{0} - F_{j-\frac{1}{2}}^{0}\,\tilde{T}_{j-\frac{1}{2}}^{0} + G_{j-\frac{1}{2}}^{0}\,\tilde{T}_{j+\frac{1}{2}}^{0} + H_{j-\frac{1}{2}}^{0} = 0. \tag{3.30}$$

Comparing this with eq. (3.9), we see that

$$\Psi_{j-\frac{1}{2}}^{0} = H_{j-\frac{1}{2}}^{0} \qquad (j = 1, 2, \cdots s). \tag{3.31}$$

The system of equations (3.8) is then solved in the usual way to get the proper initial tensions $T^0_{j-\frac{1}{2}}$.

To obtain tentative values for the coordinates at $t = t^1$, we make use of their Taylor series expansions about the point $t = t^0$, namely:

$$
\begin{aligned}
x_j^1 &= x_j^0 + (\Delta t)\dot{x}_j^0 + \tfrac{1}{2}(\Delta t)^2\ddot{x}_j^0 + \cdots \\
y_j^1 &= y_j^0 + (\Delta t)\dot{y}_j^0 + \tfrac{1}{2}(\Delta t)^2\ddot{y}_j^0 + \cdots
\end{aligned} \tag{3.32}
$$

Taking $\dot{x}_j^0$ and $\dot{y}_j^0 = 0$, and substituting eqs. (3.5) for $\ddot{x}_j^0$ and $\ddot{y}_j^0$, we find

$$
\begin{aligned}
\bar{x}_j^1 &= x_j^0 + \tfrac{1}{2}\,[-\,P_j^0T^0_{j-\frac{1}{2}} + R_j^0T^0_{j+\frac{1}{2}} + U_j^0] \\
\bar{y}_j^1 &= y_j^0 + \tfrac{1}{2}\,[-\,Q_j^0T^0_{j-\frac{1}{2}} + S_j^0T^0_{j+\frac{1}{2}} + V_j^0].
\end{aligned} \tag{3.33}
$$

The corrections to the tensions are then determined by the system of equations (3.20) in the usual manner. Finally, the corrections to the coordinates are computed as follows:

$$
\begin{aligned}
\delta x_j^1 &= \tfrac{1}{2}\,[-\,P_j^0\,\delta T^0_{j-\frac{1}{2}} + R_j^0\,\delta T^0_{j+\frac{1}{2}}] \\
\delta y_j^1 &= \tfrac{1}{2}\,[-\,Q_j^0\,\delta T^0_{j-\frac{1}{2}} + S_j^0\,\delta T^0_{j+\frac{1}{2}}]
\end{aligned} \tag{3.34}
$$

and the corrected coordinates are given by:

$$
\begin{aligned}
x_j^1 &= \bar{x}_j^1 + \delta x_j^1 \\
y_j^1 &= \bar{y}_j^1 + \delta y_j^1.
\end{aligned} \tag{3.35}
$$

4. Analysis of Numerical Stability. In order to obtain a valid solution of the system of partial differential equations governing the generalized motion of a cable, it is necessary to insure the stability (in the sense discussed in [2], [3], [4]) of the equivalent finite-difference system (2.2), (2.6), (3.3), (3.4). In this section we will derive the criteria for stability of this system of equations. We will also show that whereas the system of finite-difference equations (2.2), (2.6), (3.3), (3.4) is stable for sufficiently small time intervals Δt, the system (2.2), (3.2), (3.3), (3.4) is always unstable. This characteristic of the latter system has led to the abandonment of this simpler set of equations in favor of the more difficult nonlinear system (2.2), (2.6), (3.3), (3.4).

In order to determine the stability of a system of finite-difference equations, we study the growth of a small disturbance or perturbation. The conditions for stability are said to be satisfied if the amplitude of a small disturbance, introduced at any time, t, in any of the dependent variables, does not increase exponentially with successive time steps. This condition may be stated as follows:

If $\delta F(s, t)$ and $\delta F(s, t + \Delta t)$ are values of a variation (or perturbation) in any of the dependent variables x, y, T in the system, then it is said to be stable provided $|\,\delta F(s,\, t + \Delta t)/\delta F(s,\, t)\,| \leqq 1$. We introduce perturbations δx, δy, δT in the dependent variables x, y, T, respectively. To simplify the stability investigation, we shall omit the terms pertaining to the suspended weights as well as all terms involving virtual inertia. From eqs. (2.2), (2.6), (3.3), and (3.4) we obtain the following variational system of equations:

$$
\begin{aligned}
m_j\delta\ddot{x}_j = T_{j+\frac{1}{2}}\,\delta\cos\theta_{j+\frac{1}{2}} - T_{j-\frac{1}{2}}\,\delta\cos\theta_{j-\frac{1}{2}} + \cos\theta_{j+\frac{1}{2}}\,\delta T_{j+\frac{1}{2}} \\
- \cos\theta_{j-\frac{1}{2}}\,\delta T_{j-\frac{1}{2}} - \tfrac{1}{2}[D_{j+\frac{1}{2}}\,\delta\sin\theta_{j+\frac{1}{2}} + D_{j-\frac{1}{2}}\,\delta\sin\theta_{j-\frac{1}{2}} \\
+ \sin\theta_{j+\frac{1}{2}}\,\delta D_{j+\frac{1}{2}} + \sin\theta_{j-\frac{1}{2}}\,\delta D_{j-\frac{1}{2}}],
\end{aligned}
$$

$$(4.1)\quad m_j\delta\ddot{y}_j = T_{j+\frac{1}{2}}\,\delta\sin\theta_{j+\frac{1}{2}} - T_{j-\frac{1}{2}}\,\delta\sin\theta_{j-\frac{1}{2}} + \sin\theta_{j+\frac{1}{2}}\,\delta T_{j+\frac{1}{2}}$$
$$- \sin\theta_{j-\frac{1}{2}}\,\delta T_{j-\frac{1}{2}} + \tfrac{1}{2}[D_{j+\frac{1}{2}}\,\delta\cos\theta_{j+\frac{1}{2}} + D_{j-\frac{1}{2}}\,\delta\cos\theta_{j-\frac{1}{2}}$$
$$+ \cos\theta_{j+\frac{1}{2}}\,\delta D_{j+\frac{1}{2}} + \cos\theta_{j-\frac{1}{2}}\,\delta D_{j-\frac{1}{2}}],$$
$$\cos\theta_{j+\frac{1}{2}}\,\delta\cos\theta_{j+\frac{1}{2}} + \sin\theta_{j+\frac{1}{2}}\,\delta\sin\theta_{j+\frac{1}{2}} = 0,$$

where

$$\delta D_{j+\frac{1}{2}} = -2f^D_{j+\frac{1}{2}}\,|\,q_{j+\frac{1}{2}}\,|\,\delta q_{j+\frac{1}{2}}\,,$$
$$\delta q_{j+\frac{1}{2}} = -\tfrac{1}{2}[(\dot{x}_{j+1} - c) + (\dot{x}_j - c)]\,\delta\sin\theta_{j+\frac{1}{2}} + \tfrac{1}{2}(\dot{y}_{j+1} + \dot{y}_j)\,\delta\cos\theta_{j+\frac{1}{2}}$$
$$- \tfrac{1}{2}\sin\theta_{j+\frac{1}{2}}(\delta\dot{x}_{j+1} + \delta\dot{x}_j) + \tfrac{1}{2}\cos\theta_{j+\frac{1}{2}}\,(\delta\dot{y}_{j+1} + \delta\dot{y}_j);$$

and where

$$\delta\cos\theta_{j+\frac{1}{2}} = (\delta x_{j+1} - \delta x_j)/l_{j+\frac{1}{2}}\,,\qquad \delta\sin\theta_{j+\frac{1}{2}} = (\delta y_{j+1} - \delta y_j)/l_{j+\frac{1}{2}}\,;$$
$$\delta\dot{x}_j^{n-\frac{1}{2}} = (\delta x_j^{\,n} - \delta x_j^{n-1})/\Delta t,\qquad \delta\dot{y}_j^{n-\frac{1}{2}} = (\delta y_j^{\,n} - \delta y_j^{n-1})/\Delta t;$$
$$\delta\ddot{x}_j^{\,n} = (\delta x_j^{n+1} - 2\delta x_j^{\,n} + \delta x_j^{n-1})/(\Delta t)^2,\qquad \delta\ddot{y}_j^{\,n} = (\delta y_j^{n+1} - 2\delta y_j^{\,n} + \delta y_j^{n-1})/(\Delta t)^2.$$

We will assume in this analysis that within a small region in the (s, t) plane the coefficients ($T_j^{\,n}$, $\cos\theta_j^{\,n}$, $D_j^{\,n}$, etc.) of the variational functions vary only slightly and hence may be treated as constants. We will denote these simply by T, $\cos\theta$, D, etc., omitting the indices. A solution of the system of equations (4.1) can then be obtained in the form

$$\delta x_j^{\,n} = a\,e^{i\beta j + \alpha n\Delta t}$$
$$\delta y_j^{\,n} = b\,e^{i\beta j + \alpha n\Delta t}$$
$$\delta T_j^{\,n} = c\,e^{i\beta j + \alpha n\Delta t}$$

where a, b, c are real constants and α complex. Substituting in eq. (4.1), we obtain a system of linear homogeneous equations for the quantities a, b and c which has a non-trival solution, provided the determinant of the coefficients is identically zero. After some algebraic simplifications, the determinant of the coefficients may be written in the form

$$(4.2)\qquad \begin{vmatrix} F - A\sin\theta & D' + B\sin\theta & \cos\theta \\ -D' + A\cos\theta & F - B\cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = 0$$

where

$$A = f\,|\,q\,|\,[2i\dot{y}\sin\beta - (l\sin\theta/\Delta t)(1 + \cos\beta)(1 - \lambda^{-1})]$$
$$B = f\,|\,q\,|\,[2i(\dot{x} - c)\sin\beta - (l\cos\theta/\Delta t)(1 + \cos\beta)(1 - \lambda^{-1})]$$
$$D' = iD\sin\beta$$
$$F = ml\xi/(\Delta t)^2 + 4T\sin^2(\beta/2)$$

and where

$$\lambda = e^{\alpha\Delta t},\qquad \xi = (\lambda - 2 + \lambda^{-1}).$$

Multiplying the elements of the determinant and simplifying, we obtain

$$(4.3)\qquad A\sin\theta + B\cos\theta - F = 0$$

But,

$$A \sin \theta + B \cos \theta = f \mid q \mid [(2i \sin \beta)\, p - (l/\Delta t)(1 + \cos \beta)(1 - \lambda^{-1})]$$

where

$$p = (\dot{x} - c) \cos \theta + \dot{y} \sin \theta$$

i.e., the tangential component of the velocity of the cable (relative to the medium). Substituting in equation (4.3), we finally obtain the characteristic equation of the variational system (4.1), namely,

$$\begin{aligned} ml\lambda^2 + \{f \mid q \mid \Delta t\, [l(1 + \cos \beta) - p\Delta t(2i \sin \beta)] \\ + 4T(\Delta t)^2 \sin^2(\beta/2) - 2ml\}\lambda + [ml - f \mid q \mid l\Delta t(1 + \cos \beta)] = 0. \end{aligned} \tag{4.4}$$

Now, comparing the first and second terms of the coefficient of λ, we find that the second term is negligibly small provided $2p\Delta t \ll l$, i.e., the tangential distance traversed by the cable in one time step is very small compared with the length of the cable segment. Since this is usually the case and, at any rate, can always be satisfied by taking the time step sufficiently small, we will omit this term from our subsequent analysis.

For the case of negligible drag, i.e., $f = 0$, approximately, we obtain from eq. (4.4)

$$\lambda^2 + [4T \sin^2 (\beta/2)(\Delta t)^2/ml - 2]\lambda + 1 = 0. \tag{4.5}$$

In order for the solution to be stable, the conditions $\mid \lambda_1 \mid \leqq 1$ and $\mid \lambda_2 \mid \leqq 1$ must both be satisfied. But if λ_1 is a solution of (4.5), then $\lambda_2 = 1/\lambda_1$ is also a solution. It follows that the conditions for stability can be satisfied only if $\mid \lambda_2 \mid = \mid 1/\lambda_1 \mid = \mid \lambda_1 \mid = 1$. Now, let $\lambda_1 = \cos \gamma + i \sin \gamma$, $\lambda_2 = \cos \gamma - i \sin \gamma = 1/\lambda_1$; then, $\mid \lambda_1 + \lambda_2 \mid = \mid 2 \cos \gamma \mid \leqq 2$. Furthermore, from eq. (4.5) we have

$$\lambda_1 + \lambda_2 = 2 - \frac{4T \sin^2(\beta/2)(\Delta t)^2}{ml}.$$

We thus obtain the inequality

$$\mid 2 - 4T \sin^2(\beta/2)(\Delta t)^2/ml \mid \leqq 2,$$

or

$$T \sin^2(\beta/2)(\Delta t)^2/ml \leqq 1. \tag{4.6}$$

This requirement is tantamount to the condition

$$\Delta t \leqq \sqrt{\frac{ml}{T}} = \frac{l}{\text{velocity of transverse wave}}.$$

In the more general case, allowing for finite drag, eq. (4.4) may be reduced to (after neglecting the second term of the coefficient of λ)

$$\begin{aligned} ml\lambda^2 + [2f \mid q \mid l\Delta t \cos^2 (\beta/2) + 4T(\Delta t)^2 \sin^2(\beta/2) - 2ml]\lambda \\ + [ml - 2f \mid q \mid l\Delta t \cos^2 (\beta/2)] = 0. \end{aligned} \tag{4.7}$$

This equation is more difficult to analyze. However, it is possible to show that both $\mid \lambda_1 \mid \leqq 1$ and $\mid \lambda_2 \mid \leqq 1$, provided that we have chosen

$$\Delta t \leqq \sqrt{\frac{ml}{T}}, \quad \text{and}$$
(4.8)
$$\Delta t \leqq \frac{m}{f\,|\,q\,|}.$$

These conditions (4.8) are both necessary and sufficient to insure stability.†

We will now show that the replacement of eq. (2.6) by its differentiated form (3.2) results in an unstable system; and that, furthermore, the use of any time interval Δt, no matter how small, does not change the unstable character of the equations. It will suffice to show that this condition exists in the case when the drag is negligible, i.e., $f = 0$. The variational equation corresponding to eq. (3.2) is,

$$\begin{aligned}(x_j - x_{j-1})(\delta\ddot{x}_j - \delta\ddot{x}_{j-1}) + (\ddot{x}_j - \ddot{x}_{j-1})(\delta x_j - \delta x_{j-1})& \\ + 2(\dot{x}_j - \dot{x}_{j-1})(\delta\dot{x}_j - \delta\dot{x}_{j-1}) + (y_j - y_{j-1})(\delta\ddot{y}_j - \delta\ddot{y}_{j-1})& \\ + (\ddot{y}_j - \ddot{y}_{j-1})(\delta y_j - \delta y_{j-1}) + 2(\dot{y}_j - \dot{y}_{j-1})(\delta\dot{y}_j - \delta\dot{y}_{j-1}) &= 0.\end{aligned}$$

Substituting appropriate values for δx and δy and neglecting terms containing f, we find that the determinant equation (4.2) is replaced by

$$(4.9)\qquad \begin{vmatrix} F & 0 & \cos\theta \\ 0 & F & \sin\theta \\ \begin{matrix}(x_j - x_{j-1})\xi + (\ddot{x}_j - \ddot{x}_{j-1})(\Delta t)^2 \\ + 2(\dot{x}_j - \dot{x}_{j-1})(1 - \lambda^{-1})\Delta t\end{matrix} & \begin{matrix}(y_j - y_{j-1})\xi + (\ddot{y}_j - \ddot{y}_{j-1})(\Delta t)^2 \\ + 2(\dot{y}_j - \dot{y}_{j-1})(1 - \lambda^{-1})\Delta t\end{matrix} & 0 \end{vmatrix} = 0.$$

Multiplying the elements of the determinant, we obtain

$$(4.10)\qquad \begin{aligned}&F\cos\theta[(x_j - x_{j-1})\xi + (\ddot{x}_j - \ddot{x}_{j-1})(\Delta t)^2 + 2(\dot{x}_j - \dot{x}_{j-1})(1 - \lambda^{-1})\Delta t] \\ &+ F\sin\theta[(y_j - y_{j-1})\xi + (\ddot{y}_j - \ddot{y}_{j-1})(\Delta t)^2 + 2(\dot{y}_j - \dot{y}_{j-1})(1 - \lambda^{-1})\Delta t] = 0.\end{aligned}$$

Equating

$$\cos\theta = (x_j - x_{j-1})/l, \qquad \sin\theta = (y_j - y_{j-1})/l;$$

and using the relations (3.1) and (3.2), we obtain in place of eq. (4.10)

$$(4.11)\qquad F\{l^2\xi - [(\dot{x}_j - \dot{x}_{j-1})^2 + (\dot{y}_j - \dot{y}_{j-1})^2](\Delta t)^2\} = 0.$$

In this case the characteristic equation is a quartic, and it is necessary that the absolute values of all four roots be $\leqq 1$ to insure the numerical stability of the system. Thus, we must examine the roots of both factors of eq. (4.11), namely,

$$(4.12)\qquad F = 0$$

and

$$(4.13)\qquad l^2\xi - [(\dot{x}_j - \dot{x}_{j-1})^2 + (\dot{y}_j - \dot{y}_{j-1})^2](\Delta t)^2 = 0.$$

It can be shown that eq. (4.12) is equivalent to the criterion (4.6) which can

† When the terms for the virtual inertia and suspended weights are included in the stability analysis, it is found that the quantity m in eq. (4.8) should be replaced by the expression $m + m^* + e + \rho(V^X \sin^2\theta + V^Y \cos^2\theta)$.

be satisfied provided

$$\Delta t \leqq \sqrt{\frac{ml}{T}}.$$

However, the pair of roots associated with the second factor of (4.11) cannot satisfy the stability condition for any finite Δt because eq. (4.13) requires that

$$\xi = [(\dot{x}_j - \dot{x}_{j-1})^2 + (\dot{y}_j - \dot{y}_{j-1})^2](\Delta t)^2/l^2,$$

an inherently positive quantity. This conclusion follows as a result of the definition $\xi = \lambda - 2 + \lambda^{-1}$. If λ_3 is a root of equation (4.13), then $\lambda_4 = 1/\lambda_3$ is also a root of this equation. As before it follows that for stability $|\lambda_3| \leqq 1$ and $|\lambda_4| = |1/\lambda_3| \leqq 1$. Hence, $|\lambda_3| = |\lambda_4| = 1$. Let $\lambda_3 = \cos\gamma + i\sin\gamma$, $\lambda_4 = \cos\gamma - i\sin\gamma = 1/\lambda_3$; then $\xi = 2(\cos\gamma - 1)$, or $-4 \leqq \xi \leqq 0$. Thus, to satisfy the stability requirement ξ must lie between 0 and -4, and consequently can never be positive.

5. Numerical Results. A number of solutions were carried out with the anchor end of the line fixed and with the surface end forced to follow the motion of trochoidal waves of varying amplitudes and periods. Several typical solutions are reproduced here for the information of the reader. In Figures 3 and 4, plots are given of the maximum tension attained along the cable as a function of time for wave heights of 6 feet and periods of 12.5 seconds and 5 seconds, respectively. The periods of the variation in maximum tension correspond to the periods of the forced vibration, as expected. The maximum tension, however, increases in magnitude from 33,250 lbs in the case of waves of 12.5 sec period to 49,500 lbs when the period is 5 seconds. In Figure 5, the maximum tension attained for wave heights of 9 ft and a period of 7.5 seconds is plotted. The maximum tension is approximately 60,000 lbs as compared with 38,500 lbs for the case of 6-ft waves with the same period.

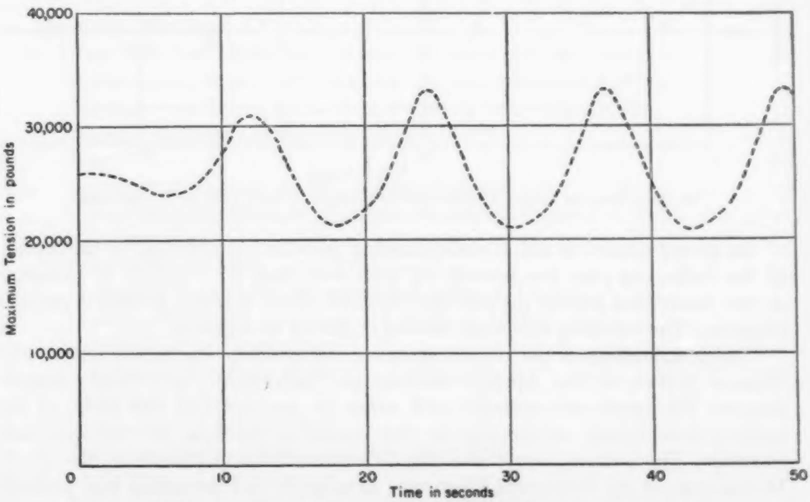

FIG. 3.—Mooring line oscillations: wave height, 6 feet; period, 12.5 seconds.

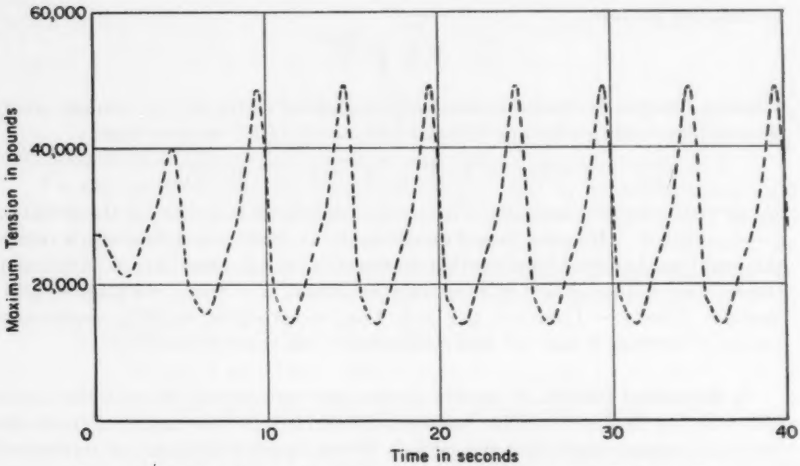

FIG. 4.—Mooring line oscillations: wave height, 6 feet; period, 5 seconds.

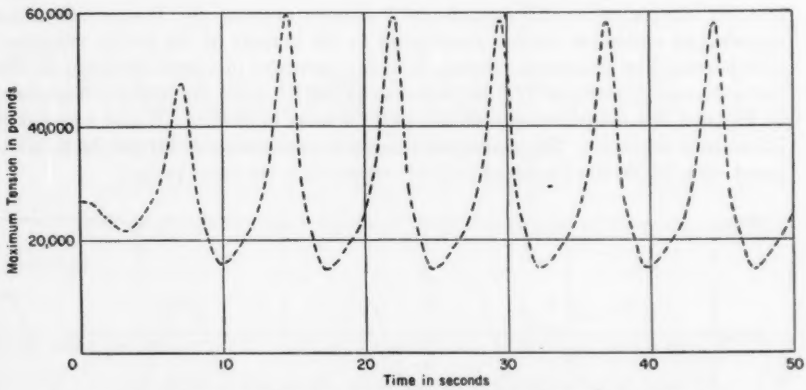

FIG. 5.—Mooring line oscillations: wave height, 9 feet; period, 7.5 seconds.

As an experiment to aid in understanding the effect of the drag on the motion of the cable, one case was carried out with zero drag (i.e., motion in vacuum). A very interesting motion pattern was obtained which does not possess a periodic character. The resulting maximum tension is plotted in Figure 6.

The programming of the various phases of this problem was carried out by Mr. Thomas McFee, of the Applied Mathematics Laboratory, in a most effective manner. The speed and accuracy with which he accomplished this phase of the solution were largely responsible for the success in meeting the required time schedules. The authors would also like to express their gratitude to Mr. R. T. McGoldrick, of the Structural Mechanics Laboratory, for proposing this problem and for a number of helpful discussions; to Dr. R. Bart, Structural Mechanics

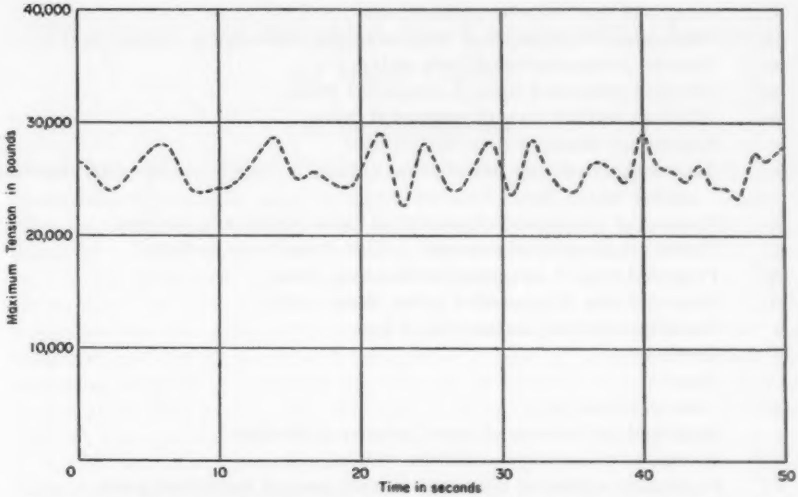

FIG. 6.—Mooring line oscillations without drag: wave height, 6 feet; period, 7.5 seconds.

Laboratory, for a number of ideas used in setting up the numerical procedure; to Dr. E. H. Kennard, David Taylor Model Basin, and Dr. R. M. Langer, Bureau of Ships, for helpful discussions in connection with the definition of the problem; to Dr. Daniel Shanks, Applied Mathematics Laboratory, for valuable suggestions; and to Miss Corinne Lundgren, Applied Mathematics Laboratory, for assistance in the preparation of the figures.

Appendix—Notation.

$C^D_{j+\frac{1}{2}}$ Drag coefficient for segment of cable between stations j and $j + 1$
C_j^X Resistance coefficient for horizontal motion of suspended prism
C_j^Y Resistance coefficient for vertical motion of suspended prism
c Velocity of uniform horizontal current
D Drag
$d_{j+\frac{1}{2}}$ Diameter of segment of cable between stations j and $j + 1$
$e_{j+\frac{1}{2}}$ Virtual mass of entrained fluid between stations j and $j + 1$
F Resultant force
$f^D_{j+\frac{1}{2}}$ Drag factor for cable $= (\rho/2)C^D_{j+\frac{1}{2}}l_{j+\frac{1}{2}}d_{j+\frac{1}{2}}$
f_j^X Horizontal drag factor for suspended prism $= (\rho/2)C_j^X S_j^X$
f_j^Y Vertical drag factor for suspended prism $= (\rho/2)C_j^Y S_j^Y$
g Acceleration due to gravity
I_j Component of inertia tensor
i Imaginary unit
J_j Component of inertia tensor
j Subscript denoting station number along line
K_j Component of inertia tensor
$k_{j+\frac{1}{2}}$ Virtual inertia coefficient for segment of cable between stations j and $j + 1$

$l_{j+\frac{1}{2}}$ Length of line between stations j and $j + 1$
m_j Mean mass of segments of cable adjoining station j
m_j^* Mass of prism suspended from station j
m_j^X Effective horizontal mass of suspended prism
m_j^Y Effective vertical mass of suspended prism
n Superscript denoting time-step number
o Superscript denoting initial state (origin in time), or subscript denoting anchor end of line
p Tangential component of velocity of cable (relative to medium)
q Normal component of velocity of cable (relative to medium)
S_j^X Projected area of suspended prism along x-axis
S_j^Y Projected area of suspended prism along y-axis
s Subscript denoting surface end of line
T Tension
t Time
Δt Time-step interval
u Magnitude of velocity of cable (relative to medium)
V_j^* Volume of prism suspended from station j
V_j^X Equivalent volume of horizontal virtual mass of suspended prism
V_j^Y Equivalent volume of vertical virtual mass of suspended prism
W_j Mean net weight of segments of cable adjoining station j
W_j^* Net weight of prism suspended from station j
X Horizontal component of resultant external force
X_j^* Horizontal component of damping force on suspended prism
x Horizontal coordinate of cable
Y Vertical component of resultant external force
Y_j^* Vertical component of damping force on suspended prism
y Vertical coordinate of cable
α Damping coefficient of the perturbation functions
β Angular wave number of the perturbation functions
δ The variation of
θ Angle between horizontal and tangent to cable
λ Eigenvalue (root of characteristic equation)
$\mu_{j+\frac{1}{2}}$ Linear density of segment of cable between stations j and $j + 1$
ρ Density of fluid medium
$\sigma_{j+\frac{1}{2}}$ Cross-section area of segment of cable between stations j and $j + 1$
$\cdot$ Dot signifies differentiation with respect to time
$\sim$ Tilde signifies tentative value of a variable

Applied Mathematics Laboratory
David Taylor Model Basin
Washington 7, District of Columbia

1. T. S. Walton & H. Polachek, "Calculation of nonlinear transient motion of cables," David Taylor Model Basin Report 1279, 1959.
2. G. G. O'Brien, M. A. Hyman & S. Kaplan, "A study of the numerical solution of partial differential equations," *Jn. Math. and Phys.*, v. 29, 1951, p. 223–251.
3. P. D. Lax & R. D. Richtmeyer, "Stability of difference equations," *Com. Pure Appl. Math.*, v. 9, 1956, p. 267–293.
4. G. Ludford, H. Polachek & R. J. Seeger, "On unsteady flow of compressible viscous fluids," *Jn. Appl. Phys.*, v. 24, 1953, p. 490–495.

Abscissas and Weights for Lobatto Quadrature of High Order

By Philip Rabinowitz

In recent years, Gaussian quadrature has become the standard method for numerical integration in many computer installations [1, 2]. In general, Gaussian rules are most economical since an n-point rule is exact for polynomials up to degree $2n - 1$ and no rule can do better. However, for particular classes of functions and for particular applications, other rules may be more efficient. Thus there are occasions when we prefer a closed rule, i.e. one which includes among its abscissas the two end points of the integration interval. This is the case when the integrand vanishes at the two end points as it does in Longman's method for evaluating integrals of oscillating functions [3]. If we want to check a quadrature over a given interval by doing two additional quadratures, each over half the interval, then a closed rule will save at least two evaluations of the integrand.

Let us normalize our integration interval to $(-1, 1)$ and consider the closed symmetric n-point integration rule with n odd, $n = 2m + 1$,

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=-m}^{m} a_{kn} f(x_{kn}) + E_n \tag{1}$$

with $x_{\pm mn} = \pm 1$, $x_{-kn} = -x_{kn}$, and $a_{-kn} = a_{kn}$. It was first shown by Lobatto* [4] (cf [5]) that the $2n - 2$ quantities, $n - 2$ abscissas x_{kn} and n weights a_{kn}, can be so chosen that the integration is exact for polynomials up to degree $2n - 3$. The abscissas x_{kn} are the zeros of $P'_{n-1}(x)$, the first derivative of the Legendre polynomial of degree $n - 1$, $P_{n-1}(x)$, and the corresponding weights a_{kn} are given by

$$a_{kn} = \frac{2}{n(n-1)[P_{n-1}(x_{kn})]^2} \tag{2}$$

while $a_{mn} = a_{-mn} = 2/n(n - 1)$. In this case

$$E_n = -\frac{n(n-1)^3 2^{2n-1}[(n-2)!]^4}{(2n-1)[(2n-2)!]^3} f^{(2n-2)}(\xi). \tag{3}$$

Tables of abscissas and weights for Lobatto quadrature (where only the values of $k = 0, \cdots, m$ are listed) were calculated on WEIZAC to 19D for $n = 5(4)$

Received May 20, 1959; in revised form August 7, 1959.

* According to Scarborough [6], Lobatto modified Gauss's integration rule so as to include the end values and also the value of the function at the midpoint of the interval. Radau [7] extended Lobatto's method to the general case of m fixed abscissas in an n-point rule ($m \leqq n$). Kopal [8] designates by Radau quadrature all rules with m preassigned abscissas. However, Hildebrand [5] and others use Radau quadrature to denote the case where only the left-hand endpoint of the integration interval is preassigned and Lobatto quadrature, the case where both endpoints are preassigned. Radau [7] tabulated Lobatto abscissas and weights for $n = 2(1)8$ to eight decimal places and for $n = 9(1)11$ to ten decimal places and Kopal [8] has reproduced them.

25(8) 49(16) 97. Table 1 lists these values, with the exception of $n = 81, 97$. Values for $n = 81, 97$ have been deposited in the Unpublished Mathematical Tables File. The zeros of $P'_{n-1}(x)$ were computed with double precision routines using Newton's method.

TABLE 1
Abscissas and weights for Lobatto quadrature

x_{kn}	w_{kn}
$n = 5$	
.000000000000000000	.711111111111111111
.654653670707977143 8	.544444444444444444
1.000000000000000000	.100000000000000000
$n = 9$	
.000000000000000000	.371519274376417233 6
.363117463826178158 7	.346428510973046345 1
.677186279510737753 4	.274538712500161735 3
.899757995411460157 3	.165495361560805525 0
1.000000000000000000	.027777777777777777 8
$n = 13$	
.000000000000000000	.251930849333446736 0
.249286930106239992 6	.244015790306676356 5
.482909821091336201 7	.220767793566110086 1
.686188469081757426 1	.183646865203550092 0
.846347564651872316 9	.134981926689608349 1
.953309846642163911 9	.077801686746818927 8
1.000000000000000000	.012820512820512820 5
$n = 17$	
.000000000000000000	.190661874753469433 3
.189511973518317388 3	.187216339677619235 9
.372174433565477041 9	.177004253515657870 4
.541385399330101539 1	.160394661997621539 5
.691028980627684705 4	.137987746201926559 1
.815696251221770307 1	.110592909007028161 4
.910879995915573595 6	.079198270503687119 2
.973132176631418314 2	.044921940543254209 6
1.000000000000000000	.007352941176470588 2
$n = 21$	
.000000000000000000	.153385190332174948 5
.152785515802185466 0	.151587575111681384 5
.301989856508764887 3	.146236862447977459 3
.444115783279002101 2	.137458462860041343 6
.575831960261830686 9	.125458121190868948 0
.694051026062223232 6	.110517083219123335 3
.796001926077712404 7	.092985467957886065 3
.879294755323590464 4	.073273918185074144 2
.941976296959745534 3	.051843169000849625 1
.982572296604548028 2	.029184840098505458 6
1.000000000000000000	.004761904761904761 9
$n = 25$	
.000000000000000000	.128308389298661928 3
.127957059483106972 7	.127254977538331447 0
.253813064168876580 2	.124112038937950290 7
.375501457859227233 2	.118931179406811825 4
.491024114818878382 6	.111797466268320888 2
.598484147279993268 1	.102828030347957830 8
.696117048815134366 8	.092170139910620421 9
.782319659240716780 4	.079998774836292981 8
.855676465835316577 5	.066513728675312784 7
.914982770734622578 3	.051936228368491474 6
.959264138252534478 9	.036504738794271372 0
.987789944931493709 3	.020465168932974385 3
1.000000000000000000	.003333333333333333 3

Table 1—*Continued*

x_{kn}		a_{kn}
	$n = 33$	
.0000000000000000000		.0966987101027165590
.0965481881761070063		.0962472849729854620
.1921949314674772257		.0948972243945918158
.2860472014876740416		.0926611334422414635
.3772287242533936350		.0895598897470774007
.4648881616321067560		.0856224485318131325
.5482070599191116231		.0808855721934550922
.6264074912812682573		.0753934869239738285
.6987593166181625957		.0691974694940161476
.7645870017935286280		.0623553678524653054
.8232759230040674696		.0549310594426269679
.8742781007505622206		.0469938504610241705
.9171173034509412408		.0386178147718139676
.9513934513969957434		.0298810459167464775
.9767861633169063015		.0208646090176033601
.9930563584336583437		.0116484483922677346
1.0000000000000000000		.0018939393939393939
	$n = 41$	
.0000000000000000000		.0775879240384808316
.0775101379362542756		.0773546125373055485
.1545541206838400005		.0766560811933174706
.2306685965595302588		.0754965310459179728
.3053958040294457421		.0738829357473896064
.3782863247310832047		.0718249996197639705
.4489017863148284990		.0693350992870948951
.5168174988471293170		.0664282092328427060
.5816250089152126880		.0631218117275497954
.6429345560645854229		.0594357916635436557
.7003774167806829578		.0553923169189143105
.7536081218869513649		.0510157049479865135
.8033065339580917814		.0463322763486516145
.8461797721039918422		.0413701961555726408
.8849639721696151228		.0361593034446895125
.9184258706979038132		.0307309290979224586
.9463641996474921212		.0251176983936010865
.9686108695676818301		.0193532985662516799
.9850318591247247763		.0134720710944706066
.9955271274452843615		.0075066587628515885
1.0000000000000000000		.0012195121951219512
	$n = 49$	
.0000000000000000000		.0647854331007237352
.0647401379908991291		.0646495667764058446
.1292087333163799330		.0642425376733883324
.1931353822540207100		.0635660530111866414
.2562519541901461885		.0626229501980326849
.3182937162511015596		.0614171849294469270
.3790004436827688739		.0599538145961051888
.4381175113207418406		.0582389770704910239
.4953969615727524004		.0562798649611358469
.5505985444332280122		.0540846954421352817
.6034907251667665006		.0516626757839646535
.6538516554322124686		.0490239647292254072
.7014701037710529541		.0461796298735572735
.7461463415517796826		.0431416012270164702
.7876929806436540091		.0399226211435904434
.8259357592850707376		.0365361908133766348
.8607142728047894851		.0329965135057250782
.8918826460264424271		.0293184347098430027
.9193101442725844907		.0255173791671367047
.9428817196631997580		.0216092842408220077
.9624984879932527299		.0176105268167831063
.9780781245013720159		.0135378293815818172
.9895551246192307357		.0094080469628555945
.9968804559100229277		.0052366002997777452
1.0000000000000000000		.0008503401360544218

TABLE 1—*Continued*

x_{kn}	a_{kn}
$n = 65$	
.0000000000000000000	.0487112532912551590
.0486919954825551174	.0486534844338217974
.0972684989276217033	.0484803148828198298
.1456142922319649050	.0481921553771416351
.1936147045111101818	.0477896893990385782
.2411558840858965205	.0472738715529625068
.2881250685259474757	.0466459253013279525
.3344108521095384325	.0459073400625635379
.3799034500654639962	.0450598676783326357
.4244949589701384082	.0441055182582980979
.4680796126822756626	.0430465554122800447
.5105540332080720304	.0418854908811053505
.5518174759018253800	.0406250785788684140
.5917720684203420792	.0392683080607104407
.6303230428642674229	.0378183974315733463
.6673789605555873413	.0362787857126874349
.7028519289179370194	.0346531246837989863
.7366578099449523957	.0329452702203182977
.7687164197616275394	.0311592731456411222
.7989517188043673806	.0292993696198098245
.8272919921669404206	.0273699710863165681
.8536700196814493087	.0253756537989475783
.8780232353249457100	.0233211479495077525
.9002938755616070109	.0212113264134911180
.9204291162429907785	.0190511931200282741
.9383811976826655570	.0168458710220649483
.9541075374600222128	.0146005895504222135
.9675708302669136143	.0123206711194442981
.9787391331919988895	.0100115149738883044
.9875859307695091355	.0076785701335481211
.9940901501184231212	.0053272417215879641
.9982358589851681587	.0029620325412562160
1.0000000000000000000	.0004807692307692308

$$x_{kn}^{(i+1)} = x_{kn}^{(i)} - \frac{P'_{n-1}(x_{kn}^{(i)})}{P''_{n-1}(x_{kn}^{(i)})} = x_{kn}^{(i)} - \Delta. \tag{4}$$

Since [9]

$$(1 - x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)] \tag{5}$$

and

$$(1 - x^2)P''_n(x) = 2xP'_n(x) - n(n+1)P_n(x), \tag{6}$$

Δ reduced to the following formula which only involves Legendre polynomials:

$$\Delta = \frac{P_{n-2}(x_{kn}^{(i)}) - x_{kn}^{(i)}P_{n-1}(x_{kn}^{(i)})}{2x_{kn}^{(i)}\,[P_{n-2}(x_{kn}^{(i)}) - x_{kn}^{(i)}\,P_{n-1}(x_{kn}^{(i)})] - nP_{n-1}(x_{kn}^{(i)})}. \tag{7}$$

The iteration was terminated when $|\Delta|$ became smaller than 2^{-74}. The initial values $x_{kn}^{(0)}$ were taken to be the arithmetic means of two successive zeros of $P_{n-1}(x)$ tabulated by Davis and Rabinowitz. [10, 11] With this choice of initial value, convergence was achieved in at most six iterations.

The values in the table were checked on the computer by calculating the sums $\sum_{k=-m}^{m} a_{kn}x_{kn}^r$ for $r = 0, 2, 4, 8, 16, 32$. For n such that $r \leqq 2n - 3$, these sums were equal to $(2/r + 1)$ to 19 decimal places. In addition $\sum_{k=-m}^{m} a_{kn}$ was computed

by hand from the final manuscript and was found to deviate from the correct value 2 by at most 4 units in the 19th decimal place. This accuracy is acceptable since we were summing rounded numbers. A further check was made on the computer by calculating $\sum_{k=1}^{m} x_{kn}^2$ and $\prod_{k=1}^{m} x_{kn}$. Each sum checked to 19 decimal places with the true value $(n-2)(n-3)/2(2n-3)$ while the square of each product checked to at least ten significant figures with the true value

$$\frac{\binom{n+1}{m+1}(m+1)}{2\binom{2n-2}{n-1}}.$$

These two identities are derived as follows:

$$P_{n-1}(x) = 2^{-(n-1)} \sum_{t=0}^{m} (-1)^t \binom{n-1}{t}\binom{2n-2-2t}{n-1} x^{n-1-2t} \tag{8}$$

$$\begin{aligned} P'_{n-1}(x) &= 2^{-(n-1)} \sum_{t=0}^{m} (-1)^t \binom{n-1}{t}\binom{2n-2-2t}{n-1} (n-1-2t)\, x^{n-2-2t} \\ &= 2^{-(n-1)}\, x \sum_{t=0}^{m-1} (-1)^t \binom{n-1}{t}\binom{2n-2-2t}{n-1} (n-1-2t)(x^2)^{m-t-1} \\ &= x\,[a_0\,(x^2)^{m-1} + a_1(x^2)^{m-2} + \cdots + a_{m-2}\,x^2 + a_{m-1}] \\ &= a_0\, x \prod_{k=1}^{m} (x + x_{kn})(x - x_{kn}) = a_0\, x \prod_{k=1}^{m} (x^2 - x_{kn}^2). \end{aligned} \tag{9}$$

Hence

$$\sum_{k=1}^{m} x^2 = -\frac{a_1}{a_0} = \frac{\binom{n-1}{1}\binom{2n-4}{n-1}(n-3)}{\binom{n-1}{0}\binom{2n-2}{n-1}(n-1)} = \frac{(n-2)(n-3)}{2(2n-3)}$$

and

$$\prod_{k=1}^{m} x^2 = (-1)^{m-1}\frac{a_{m-1}}{a_0} = \frac{\binom{n-1}{m-1}\binom{2n-2m}{n-1}2}{\binom{n-1}{0}\binom{2n-2}{n-1}(n-1)} = \frac{m+1}{2}\,\frac{\binom{n+1}{m+1}}{\binom{2n-2}{n-1}}.$$

Department of Applied Mathematics
Weizmann Institute of Science
Rehovot, Israel

1. M. W. Wilkes, D. J. Wheeler, & S. Gill, *Programs for an Electronic Digital Computer*, 2nd Edition, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1957, p. 86.

2. W. L. Frank, "Mathematical subroutines for the UNIVAC scientific computer," *Computers and Automation*, v. 6, No. 9, 1957, p. 10–13.

3. I. M. Longman, "Note on a method for computing infinite integrals of oscillatory functions," Cambridge Phil. Soc., *Proc.*, v. 52, 1956, p. 764.

4. R. Lobatto, *Lessen over de Integraal-Rekening*, The Hague, 1852, p. 207–210.

5. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill Book Co., New York, 1956, p. 334.

6. J. B. Scarborough, *Numerical Mathematical Analysis*, The Johns Hopkins Press, Baltimore, Maryland, 1950, p. 152.

7. R. RADAU, "Étude sur les formules d'approximation qui servent à calculer la valeur d'une integrale definie," *J. Math. Pures Appl.*, v. 6, No. 3, 1880, p. 283.
8. Z. KOPAL, *Numerical Analysis*, John Wiley & Sons, Inc., New York, 1955, p. 390.
9. A. ERDELYI, ET AL., *Higher Transcendental Functions*, v. 2, McGraw-Hill Book Co., New York, 1953, p. 179.
10. P. DAVIS & P. RABINOWITZ, "Abscissas and weights for Gaussian quadratures of high order," NBS, *Jn. of Research*, v. 56, 1956, p. 35.
11. P. DAVIS & P. RABINOWITZ, "Additional abscissas and weights for Gaussian quadratures of high order: values for $n = 64$, 80 and 96," NBS, *Jn. of Research*, v. 60, 1958, p. 613.

A Method for the Numerical Evaluation of Finite Integrals of Oscillatory Functions

By I. M. Longman

1. Introduction. In two previous publications [1, 2] the author has demonstrated a method, based on Euler's transformation of slowly convergent alternating series, for the numerical evaluation of infinite integrals of oscillatory functions; this can be used in many cases by a double application for the evaluation of finite integrals of oscillatory functions. For example the integral

$$\int_b^a f(x)\,dx \tag{1}$$

where $f(x)$ may have a very large number of oscillations in the range of integration can conveniently be evaluated as

$$\int_a^\infty f(x)\,dx - \int_b^\infty f(x)\,dx. \tag{2}$$

However in physical problems the finiteness of the range of integration is often associated with a kind of natural boundary of $f(x)$, such that it is impossible to extend $f(x)$ to values of x beyond the upper limit b while preserving the general character of $f(x)$. Analytically speaking, $x = b$ may be a branch point of $f(x)$. Alternatively, it may be possible to extend the range of integration to infinity as in equation (2), but the infinite integrals may not converge. As an example of the branch point difficulty we can consider the integral

$$I = \int_0^a (a^2 - x^2)^{\frac{1}{2}} \sin x\,dx, \tag{3}$$

which, if a is large, would be very difficult to compute by straightforward numerical integration owing to the large number of oscillations, and we clearly cannot apply the method of equation (2). An instance of this kind of difficulty in a physical problem is given in Pekeris [3] where the solution of a problem of sound propagation in a layered liquid is given in the form of infinite integrals where the character of the oscillatory integrand changes abruptly at a certain point in the interval of integration.

The present paper gives an extension of Euler's transformation and applies it to the numerical computation of integrals such as (3).

2. Description of the Method. For simplicity let us start by considering alternating series, and suppose that we wish to calculate the sum of a series of the form

$$S = v_0 - v_1 + v_2 - \cdots + (-1)^n v_n \tag{4}$$

Received August 13, 1959. Institute of Geophysics Publication No. 139.

where the v_i's are all positive and slowly decrease in numerical value as i increases from 0 to n. For example such a series might be

$$S = 9999^{\frac{1}{2}} - 9998^{\frac{1}{2}} + 9997^{\frac{1}{2}} - \cdots + 1^{\frac{1}{2}}. \tag{5}$$

In order to transform (4) into a form convenient for computation let us consider the associated power series,

$$S(x) = v_0 - v_1 x + v_2 x^2 - \cdots + (-1)^n v_n x^n, \tag{6}$$

which reduces to (4) when $x = 1$. If we multiply by x and add we obtain

$$\begin{aligned}(1 + x)S(x) &= v_0 - (v_1 - v_0)x \\ &\quad + (v_2 - v_1)x^2 - \cdots + (-1)^n(v_n - v_{n-1})x^n + (-1)^n v_n x^{n+1},\end{aligned}$$

or, with the usual notation for differences,

$$\Delta v_i = v_{i+1} - v_i$$

$$\Delta^{r+1} v_i = \Delta^r v_{i+1} - \Delta^r v_i\,,$$

we have

$$(1 + x)S(x) = v_0 - (\Delta v_0)x + (\Delta v_1)x^2 - \cdots + (-1)^n(\Delta v_{n-1})x^n + (-1)^n v_n x^{n+1}.$$

This gives

$$\begin{aligned}S(x) &= \frac{v_0 + (-1)^n v_n x^{n+1}}{1 + x} \\ &\quad - y[\Delta v_0 - (\Delta v_1)x + (\Delta v_2)x^2 - \cdots + (-1)^{n-1}(\Delta v_{n-1})x^{n-1}],\end{aligned} \tag{7}$$

where $y = x/(1 + x)$. A second application of this transformation to the series in square brackets in (7) yields

$$\begin{aligned}S &= \frac{v_0 + (-1)^n v_n x^{n+1}}{1 + x} - \frac{\Delta v_0 + (-1)^{n-1}(\Delta v_{n-1})x^n}{1 + x}\, y \\ &\quad + y^2[\Delta^2 v_0 - (\Delta^2 v_1)x + (\Delta^2 v_2)x^2 - \cdots + (-1)^{n-2}(\Delta^2 v_{n-2})x^{n-2}],\end{aligned}$$

and p applications give

$$\begin{aligned}S(x) &= (v_0 - y\Delta v_0 + y^2\Delta^2 v_0 - \cdots + (-1)^{p-1} y^{p-1}\Delta^{p-1} v_0)/(1 + x) \\ &\quad + (-1)^n[v_n x^{n+1} + (\Delta v_{n-1})x^n y + (\Delta^2 v_{n-2})x^{n-1}y^2 + \cdots \\ &\qquad + (\Delta^{p-1} v_{n-p+1})x^{n-p+2}y^{p-1}]/(1 + x) \\ &\quad + (-1)^p y^p[\Delta^p v_0 - (\Delta^p v_1)x + (\Delta^p v_2)x^2 - \cdots \\ &\qquad + (-1)^{n-p}(\Delta^p v_{n-p})x^{n-p}], \qquad p \leqq n.\end{aligned}$$

Putting $x = 1$ we have as a transformed form of (4)

$$\begin{aligned}S &= [(1/2)v_0 - (1/4)\Delta v_0 + (1/8)\Delta^2 v_0 - \cdots + (-1)^{p-1}2^{-p}(\Delta^{p-1} v_0)] \\ &\quad + (-1)^n[(1/2)v_n + (1/4)\Delta v_{n-1} + (1/8)\Delta^2 v_{n-2} + \cdots + 2^{-p}\Delta^{p-1} v_{n-p+1}] \\ &\quad + 2^{-p}(-1)^p[\Delta^p v_0 - \Delta^p v_1 + \Delta^p v_2 - \cdots + (-1)^{n-p}\Delta^p v_{n-p}], \qquad p \leqq n.\end{aligned} \tag{8}$$

This result is of course exact, but for large values of p (assuming that the high order differences are small) the later terms in the first two square brackets and the whole of the third square bracket in equation (8) can be neglected since 2^{-p} will be negligible.

Assuming then, that n is large, we have for (4) the excellent approximation

$$\begin{aligned} S = (1/2)v_0 - (1/4)\Delta v_0 + (1/8)\Delta^2 v_0 - \cdots \\ + (-1)^n[(1/2)v_n + (1/4)\Delta v_{n-1} + (1/8)\Delta^2 v_{n-2} + \cdots] \end{aligned} \tag{9}$$

which represents a kind of double application of Euler's transformation.

3. Examples. The utility of the series (9) will be demonstrated by a number of examples.

1. Consider the series

$$S = 1 - 1/2 + 1/3 - 1/4 + \cdots - 1/1000. \tag{10}$$

To evaluate (10) by means of equation (9) we split off the first eight terms (whose differences do not decrease very rapidly) and evaluate

$$S' = 1/9 - 1/10 + 1/11 - \cdots - 1/1000. \tag{11}$$

The contribution at the first eight terms of S is 0.634524, and so

$$S = 0.634524 + S'.$$

The differencing of the first few and the last few terms of S' is shown in Table 1. From this, by means of equation (9), we obtain

$$\begin{aligned} S' = (1/2) \times 0.111111 + (1/4) \times 0.011111 + (1/8) \times 0.002020 \\ + (1/16) \times 0.000505 + (1/32) \times 0.000156 + (1/64) \times 0.000057 \\ - (1/2) \times 0.001000 + (1/4) \times 0.000001 = 0.058123. \end{aligned}$$

Thus $S = 0.692647$. As a check we can calculate (10) as the difference between two infinite series

$$\begin{aligned} S &= (1 - 1/2 + 1/3 - 1/4 + \cdots) \\ &\qquad - (1/1001 - 1/1002 + 1/1003 - \cdots) \\ &= \ln 2 - (1/1001 - 1/1002 + 1/1003 - \cdots). \end{aligned} \tag{12}$$

The series in parentheses in (12) can be evaluated by applying Euler's transformation, and its sum is easily shown to be

$$0.000499.$$

Thus working with equation (12) we obtain

$$S = 0.693147 - 0.000499 = 0.692648,$$

agreeing with our previous result.

2. In example 1 we were able to extend our series to infinity so that we really had no need to use the transformation (9). We now consider, however, an example

Table 1

		Δ^2		Δ^4		Δ^6
$9^{-1} = 0.111111$						
	-11111					
$10^{-1} = 0.100000$		2020				
	-9091		-505			
$11^{-1} = 0.090909$		1515		156		
	-7576		-349		-57	
$12^{-1} = 0.083333$		1166		99		24
	-6410		-250		-33	
$13^{-1} = 0.076923$		916		66		
	-5454		-184			
$14^{-1} = 0.071429$		732				
	-4762					
$15^{-1} = 0.066667$						
- - -						
$994^{-1} = 0.001006$						
	-1					
$995^{-1} = 0.001005$						
	-1					
$996^{-1} = 0.001004$						
	-1					
$997^{-1} = 0.001003$						
	-1					
$998^{-1} = 0.001002$						
	-1					
$999^{-1} = 0.001001$						
	-1					
$1000^{-1} = 0.001000$						

Table 2

		Δ^2		Δ^4		
$9999^{\frac{1}{2}} = 99.995000$						
	-5000					
$9998^{\frac{1}{2}} = 99.990000$		-2				
	-5002		$+2$			
$9997^{\frac{1}{2}} = 99.984998$		0		1		
	-5002		$+3$			
$9996^{\frac{1}{2}} = 99.979996$		3				
	-4999		-6			
$9995^{\frac{1}{2}} - 99.974997$		-3				
	-5002					
$9994^{\frac{1}{2}} = 99.969995$						
- - -						
$15^{\frac{1}{2}} = 3.872983$						
	-131326					
$14^{\frac{1}{2}} = 3.741657$		-4780				
	-136106		-563			
$13^{\frac{1}{2}} = 3.605551$		-5343		-122		
	-141449		-685		-35	
$12^{\frac{1}{2}} = 3.464102$		-6028		-157		-27
	-147477		-842		-62	
$11^{\frac{1}{2}} = 3.316625$		-6870		-219		
	-154347		-1061			
$10^{\frac{1}{2}} = 3.162278$		-7931				
	-162279					
$9^{\frac{1}{2}} = 3.000000$						

of a series which cannot be so extended. Such a series is given in equation (5). This series has 9,999 terms and its direct evaluation would be a rather lengthy computation. However it is easily computed by an application of our transformation in equation (9). Differencing the first few terms, and a few terms near the end of the series we obtain Table 2. We split off the last eight terms of the series (which do not difference so well) and obtain

$$\begin{aligned} S = {} & (1/2) \times 99.995000 + (1/4) \times 0.005000 - (1/8) \times 0.000002 \\ & + (1/2) \times 3.000000 - (1/4) \times 0.162278 - (1/8) \times 0.007931 \\ & - (1/16) \times 0.001061 - (1/32) \times 0.000219 - (1/64) \times 0.000062 \\ & - 8^{1/2} + 7^{1/2} - 6^{1/2} + 5^{1/2} - 4^{1/2} + 3^{1/2} - 2^{1/2} + 1^{1/2} = 50.378853. \end{aligned}$$

3. Now let us turn to the evaluation of oscillatory integrals. We will illustrate this by evaluating the integral (3) for $a = 100\pi$. We have

$$I = \int_0^{100\pi} (100^2\pi^2 - x^2)^{1/2} \sin x \, dx. \tag{3'}$$

By splitting up the range of integration, I can be expressed as the series

$$I = \sum_{r=0}^{99} (-1)^r \int_0^{\pi} [100^2\pi^2 - (r\pi + x)^2]^{1/2} \sin x \, dx \tag{13}$$

which is of the form (4). A few integrals near the beginning and near the end of the series (13) were evaluated on an IBM 709 computer using a 16 point Gaussian integration formula. The results are tabulated and differenced in Table 3. Applying our method we obtain

$$\begin{aligned} I = {} & (1/2) \times 628.30915 + (1/4) \times 0.06285 - (1/8) \times 0.06284 \\ & + (1/16) \times 0.00004 - [(1/2) \times 325.85292 - (1/4) \times 10.14092 \\ & - (1/8) \times 0.41146 - (1/16) \times 0.03382 - (1/32) \times 0.00472 \\ & - (1/64) \times 0.00089 - (1/128) \times 0.00022] + 315.26077 \\ & - 304.17027 + 292.52472 - \cdots - 60.96022 \\ = {} & 298.43558. \end{aligned}$$

As a check we apply the method differently to Table 3, obtaining

$$\begin{aligned} I = {} & 628.30915 - 628.24630 + (1/2) \times 628.12061 - (1/4) \times 0.18857 \\ & - (1/8) \times 0.06298 - (1/16) \times 0.00002 + (1/2) \times 238.71325 \\ & - (1/4) \times 14.76162 - (1/8) \times 0.96185 - (1/16) \times 0.14230 \\ & - (1/32) \times 0.03311 - (1/64) \times 0.00997 - (1/128) \times 0.00362 \\ & - (1/256) \times 0.00150 - 222.79836 + 209.46181 - \cdots - 60.96002 \\ = {} & 298.43557 \end{aligned}$$

agreeing with our previous result.

Table 3

q	x_q		Δ^2		Δ^4		Δ^6	
1	628.30915							
		−6285						
2	628.24630		−6284					
		−12569		−4				
3	628.12061		−6288					
		−18857		−10				
4	627.93204		−6298					
		−25155		−2				
5	627.68049		−6300					
		−31455						
6	627.36594							
79	381.13726							
		−837885						
80	372.75841		−30116					
		−868001		−2211				
81	364.07840		−32327		−316			
		−900328		−2527		−67		
82	355.07512		−34854		−383		−22	
		−935182		−2910		−89		
83	345.72330		−37764		−472			
		−972946		−3382				
84	335.99384		−41146					
		−1014092						
85	325.85292							
85	325.85292							
		−1059215						
86	315.26077		−49835					
		−1109050		−5670				
87	304.17027		−55505		−1256			
		−1164555		−6926		−423		
88	292.52472		−62431		−1679		212	
		−1226986		−8605		−635		−150
89	280.25486		−71036		−2314		362	
		−1298022		−10919		−997		
90	267.27464		−81955		−3311			
		−1379977		−14230				
91	253.47487		−96185					
		−1476162						
92	238.71325							
93	222.79836							
94	205.46181							
95	186.30583							
96	164.30583							
97	139.47917							
98	108.12528							
99	60.96022							

4. Conclusion. An extension of Euler's transformation has been presented and its use in the computation of finite integrals of oscillatory functions demonstrated. It is believed that its application will render feasible the numerical solution of physical problems hitherto regarded as intractable.

5. Acknowledgments. The author desires to record his appreciation of the computing facilities made available to him at the Western Data Processing Center of the University of California at Los Angeles.

Institute of Geophysics
University of California
Los Angeles, California

1. I. M. LONGMAN, "Note on a method for computing infinite integrals of oscillatory functions," Cambridge Phil. Soc. *Proc.*, v. 52, 1956, p. 764.
2. I. M. LONGMAN, "Tables for the rapid and accurate numerical evaluation of certain infinite integrals involving Bessel functions," *MTAC*, v. 11, 1957, p. 166.
3. C. L. PEKERIS, Geol. Soc. Am. *Mem.*, No. 27, 1948, p. 45. (See equations A17, A18, A19).

Products of Laguerre Polynomials

By Joseph Gillis and George Weiss

1. Problems occasionally arise in theoretical physics where one wishes to express the product of two linear combinations of Laguerre polynomials as a linear series of these same polynomials. The purpose of this note is to investigate the coefficients C_{rst} where the following definitions are used:

$$L_n(x) = \sum_{r=0}^{n} (-1)^n \binom{n}{r} x^r/r! \tag{1}$$

$$L_r(x)L_s(x) = \sum_{t=|r-s|}^{r+s} C_{rst} L_t(x) \tag{2}$$

The limits of the sum in (2) follow, in fact, from repeated application of the recurrence formula for the L_n's. This can be written [1] in the form

$$xL_n(x) = -(n+1)L_{n+1}(x) + (2n+1)L_n(x) - nL_{n-1}(x). \tag{3}$$

It follows from the orthogonality properties of Laguerre polynomials that

$$C_{rst} = \int_0^\infty e^{-x} L_r(x)L_s(x)L_t(x)\, dx \tag{4}$$

and, in particular, is symmetric in r, s, t. A closed formula has been obtained for C_{rst} by Watson [4]. We begin by obtaining the same formula by a very simple argument. In §3 we derive a simple recurrence relation suitable for rapidly generating the coefficients as needed when working with a high speed computing machine. This will be seen to be more useful in practice than the formal expression in equation (7).

2. It is known [2] that the Laplace transform of $L_n(x)$ is $p^{-n-1}(p-1)^n$ while that of $L_h(x)L_k(x)$ is

$$\binom{h+k}{h} \frac{(p-1)^{h+k}}{p^{h+k+1}}\, {}_2F_1\left[-h, -k; -h-k; \frac{p(p-2)}{(p-1)^2}\right].$$

Hence, taking Laplace transforms of both sides of (2), we get

$$\binom{r+s}{r} p^{-r-s-1}(p-1)^{r+s}\, {}_2F_1\left[-r, -s; -r-s; \frac{p(p-2)}{(p-1)^2}\right] = \sum_t C_{rst} p^{-t-1}(p-1)^t. \tag{5}$$

With $q = 1 - p^{-1}$ we have

$$\sum_t C_{rst} q^t = \binom{r+s}{r} q^{r+s}\, {}_2F_1[-r, -s; -r-s; (2q-1)/q^2]. \tag{6}$$

Received August 27, 1959.

Comparing coefficients of q^t in (6) gives us

$$
\begin{aligned}
C_{rst} &= \sum_n \frac{(r+s-n)!}{n!(r-n)!(s-n)!}(-2)^{2n-r-s+t}\binom{n}{2n-r-s+t} \\
&= (-\tfrac{1}{2})^p \sum_n 2^{2n} \frac{(r+s-n)!}{(r-n)!(s-n)!(2n-p)!(p-n)!}
\end{aligned} \tag{7}
$$

where we have written $p = r + s - t$. This is equivalent to Watson's formula.

The limits of the sum in (7) are defined by the requirement that none of the arguments of the factorials can be negative. It is easily confirmed from this, incidentally, that there can be no terms in the sum if t lies outside the range $(|r - s|, r + s)$.

3. To establish a recurrence relation we base ourselves on the well-known result [3] that, if $u(x)$, $v(x)$ satisfy the normalized differential equations

$$
\begin{aligned}
u''(x) + I(x)u(x) &= 0 \\
v''(x) + J(x)v(x) &= 0,
\end{aligned} \tag{8}
$$

then $y = uv$ satisfies the equation

$$
\frac{d}{dx}\left[\frac{y''' + 2(I+J)y' + (I'+J')y}{I-J}\right] + (I-J)y = 0 \tag{9}
$$

provided that $I \not\equiv J$. In case $I \equiv J$, y satisfies the third order equation

$$
y''' + 4Iy' + 2I'y = 0. \tag{10}
$$

Applying this result to the differential equation satisfied by Laguerre polynomials, after normalization, we obtain the following equation for $y = L_r(x)L_s(x) (r \geqq s)$

$$
\begin{aligned}
\mathsf{D}(y) = x^2y^{(\mathrm{iv})} &+ x(5-4x)y''' + [4 + (2\sigma - 15)x + 5x^2]y'' \\
&+ [(3\sigma - 8) - 4(\sigma - 3)x - 2x^2]y' + [\delta^2 - 3(\sigma - 1) + 2(\sigma - 1)x]y = 0
\end{aligned} \tag{11}
$$

where

$$
\begin{cases} \sigma = r + s + 1 \\ \delta = r - s. \end{cases} \tag{12}
$$

In fact, equation (11) holds whether or not $r = s$. We now substitute from (2) into $x\mathsf{D}(y)$, making use of the following formulas:

$$ xL_t'' = -(1-x)L_t' - tL_t, \tag{13} $$

$$ x^2L_t''' = [2 - (t+2)x + x^2]L_t' + t(2-x)L_t, \tag{14} $$

$$
\begin{aligned}
x^3L_t^{(\mathrm{iv})} = &-\{6 - 2(2t+3)x + (2t+3)x^2 - x^3\}L_t' \\
&- t\{6 - (t+3)x + x^2\}L_t.
\end{aligned} \tag{15}
$$

Of these equations, (13) is simply the differential equation of L_t, while (14) and (15) are obtained from it by differentiation followed by substitution from (13) itself. We thus obtain, after some reduction,

$$ x\mathsf{D}(L_t) = \tau x(1-2x)L_t' + \{2\tau x^2 + x[\delta^2 - t^2 - \tau(2t+3)]\}L_t \tag{16} $$

where

$$\tau = \sigma - t - 1 = r + s - t \tag{17}$$

But, [1],

$$xL_t' = (t + 1)L_{t+1} - (t + 1 - x)L_t\,. \tag{18}$$

Substituting from (18) into (16) and making repeated use of (3) leads finally to

$$\begin{aligned} x\mathsf{D}(L_t) = {} & 2\tau(t + 1)(t + 2)L_{t+2} - (t + 1)[\tau(4t + 5) + (\delta^2 - t^2)]L_{t+1} \\ & + (2t + 1)[(t + 1)\tau + \delta^2 - t^2]L_t - t(\delta^2 - t^2)L_{t-1}\,. \end{aligned} \tag{19}$$

For fixed r, s write $C_{rst} = A_t$. It follows from (19) that

$$x\mathsf{D}(\sum_t A_tL_t) = \sum B_tL_t \tag{20}$$

where

$$\begin{aligned} B_t = {} & -(t + 1)[\delta^2 - (t + 1)^2]A_{t+1} \\ & + (2t + 1)[\delta^2 - t^2 + (t + 1)\sigma - (t + 1)^2]A_t \\ & - t[\delta^2 - (t - 1)^2 + (4t + 1)(\sigma - t)]A_{t-1} + 2t(t - 1)(\sigma - t + 1)A_{t-2}\,. \end{aligned} \tag{21}$$

We thus obtain the recurrence relation

$$\begin{aligned} (t + 1)[\delta^2 - (t + 1)^2]A_{t+1} = {} & (2t + 1)[\delta^2 - t^2 + (t + 1)(\sigma - t - 1)]A_t \\ & - t[\delta^2 - (t - 1)^2 + (4t + 1)(\sigma - t)]A_{t-1} \\ & + 2t(t - 1)(\sigma - t + 1)A_{t-2}\,. \end{aligned} \tag{22}$$

We know that $A_t = 0$ for $t < \delta$. However (22) becomes indeterminate for $t = \delta - 1$ and so A_δ has to be calculated independently. This was to have been expected since equation (11) is homogeneous. It follows immediately from (7) that

$$A_\delta = C_{r,s,r-s} = \binom{r}{s} \tag{23}$$

When working with an electronic computer it will nearly always be more efficient to use (22) and (23) than (7). All that one need store is the binomial coefficients (23) and a sub-routine for effecting (22).

4. It may be of interest to consider some values of C_{rst} for special values of r, s, t. The value of $C_{r,s,r-s}$ is given by (23) and we deduce, by means of (22), that

$$\begin{cases} C_{r,s,r-s+1} = -2s\dbinom{r}{s} \qquad \text{and} \\ C_{r,s,r-s+2} = (r - s + 1)^{-1}s[(2s - 1)(r + 1) - 2s^2]\dbinom{r}{s}. \end{cases} \tag{24}$$

We can calculate directly from (7) that

$$\begin{cases} C_{r,s,r+s} = \dbinom{r + s}{r} \\ C_{r,s,r+s-1} = -2(r + s - 1)\dbinom{r + s - 2}{r - 1} \\ C_{r,s,r+s-2} = (2rs - r - s + 1)\dbinom{r + s - 1}{r - 1}. \end{cases} \tag{25}$$

One might also draw attention to some elementary arithmetical properties of the coefficients C_{rst}. In the first place, it is clear from (7) that the sign of C_{rst} is that of $(-1)^p$ and so of $(-1)^{r+s+t}$. Again, it follows from the same formula that C_{rst} is always an integer. For the expression

$$\frac{(r+s-n)!}{(r-n)!(s-n)!\,(2n-p)!\,(p-n)!}$$

is an integer, being a multinomial coefficient. Also, the occurrence of the term $(2n-p)!$ in the denominator imposes the limitation $2n \geqq p$ on the range of n. The result is immediate. Finally, we deduce, by putting $x = 0$ in (1) and (2) that

$$\sum_t C_{rst} = 1 \tag{26}$$

and, from symmetry, that the summation in (26) could equally be taken over s or r.

5. The relation (22) is, as we have said, well adapted to machine work. It is rather complicated for hand computation and the authors are indebted to a referee who drew their attention to the alternative relation

$$(r+1)C_{r+1,s,t} = (t+1)C_{r,s,t+1} + 2(t-r)C_{r,s,t} - rC_{r-1,s,t} + tC_{r,s,t-1}. \tag{27}$$

This follows immediately from the orthogonality and recurrence relations of the Laguerre polynomials, and is very much simpler arithmetically than (22). There is no doubt that many other relations of this type could be found. However, for machine computation they would all share the disadvantage of (27), namely, the increased programming complications involved in varying two of the subscripts. They would also be slower to generate since, to arrive at a given r, s, t, one would have to progress through a much larger number of intermediate terms.

Department of Applied Mathematics
Weizmann Institute of Science
Rehovot, Israel

1. A. Erdelyi, et. al., *Higher Transcendental Functions*, v. 2, McGraw-Hill, New York, 1953, p. 188–189.
2. A. Erdelyi, et. al., *Tables of Integral Transforms*, v. 1, McGraw-Hill, New York, 1954, p. 174–175.
3. G. N. Watson, *Theory of Bessel Functions*, University Press, Cambridge, 1948, p. 146.
4. G. N. Watson, "A note on the polynomials of Hermite and Laguerre," London Math. Soc., *Jn.*, v. 13, 1938, p. 29.

The Formula-Controlled Logical Computer "Stanislaus"

By F. L. Bauer

The evaluation of a formula of propositional calculus is considerably simplified if this formula is written in the parenthesis-free notation of the Warsaw School, [1]. The Warsaw notation may be formulated in the following way:

There are symbols for operations, e.g.: N for negation; K for conjunction; A for disjunction; E for equivalence; C for implication; and symbols p, q, r, s, t for variables.

A variable is a formula. A formula preceded by the symbol N is a formula. Two juxtaposed formulas preceded by any one of the symbols, K, A, E, C are a formula. Evaluation of such a formula is done in the following way: Each of the variable symbols $p, q, r, \cdots$ has a value 0 or 1. The operation symbol acts on the value of the one or two formulas governed by it giving the value of the compound formula.

It was remarked in 1950 by H. Angstl, [2], that a mechanical evaluation of a formula, written in Warsaw notation without brackets, can be done in the following easy way: Each of the variable symbols is represented by a box with one output, the negation by a box with one input and one output, and the other operation symbols by a box with two inputs and one output. The meaning of a formula in the Warsaw notation is given by Angstl's rule: The first input of each operation symbol is to be connected with the output of the next following symbol, either variable or operation. The symbol N excepted, the second input of each operation symbol is to be connected with the first remaining free output of a symbol going from left to right. This may be demonstrated by an example, which uses Stanislaus present capacity of eleven symbols: the tautology of transitivity of the implication $[(p \rightarrow q)\ \&\ (q \rightarrow r)] \rightarrow (p \rightarrow r)$, written in Warsaw notation

$$C\,K\,C\,p\,q\,C\,q\,r\,C\,p\,r$$

and represented according to Angstl's rule by the diagram Fig. 1.

Following the Angstl rule, there is an easy way to build a mechanism for the evaluation of a logical formula, especially for a switching network which does the necessary connections of the logical units corresponding to the boxes mentioned above. Of course, these units can be built up in the usual way, e.g., with relays. But the proper switching may be done automatically by punching the formula on a keyboard.

We found building such an apparatus highly instructive on account of our interest in formula-translation, because it is a model of a computer which immediately obeys formulas in a common language. This means that, apparently in contrast to the apparatus of Kalin and Burkhart, [3], and later modifications, logical formulas are directly "written" (e.g., typed on a keyboard) rather than wired manually by connecting outputs and inputs on a switchboard.

It seems quite clear that this idea of direct formula control is not restricted to

Received February 9, 1959; in revised form May 1, 1959.

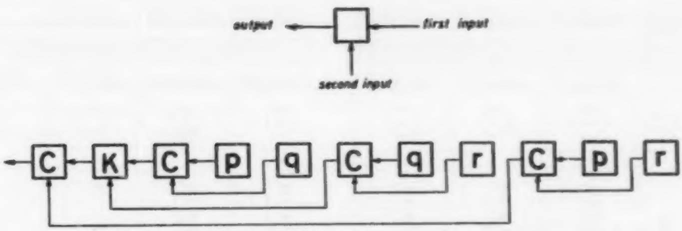

FIG. 1.—Example of a mechanical evaluation of a formula.

logical operations and the values 0, 1 only. Indeed, our interest is primarily devoted to arithmetical formulas, and we have obtained (in collaboration with Dr. K. Samelson, of Mainz) some results we believe to be remarkable. They will be published in the near future.

The wiring design of Stanislaus was done in December 1950. With the friendly help of colleagues, the toy was built in the following years, mostly from surplus material. The work suffered long interruptions, since it had a very low priority in our working program. Stanislaus was finished at the end of 1956 and presented in a lecture by H. Angstl, "Vorführung eines logistischen Rechengerätes für den Aussagenkalkül," held on January 8, 1957, in the Logistisches Seminar der Universität München. Stanislaus is now sometimes used for demonstration purposes. There might be possibilities of serious use outside the scope of our interest.

In the language of computer engineering we constructed, influenced by the material provided to us, something like a parallel-in-formula computer, but serial-in-formula operation would have been possible too, and the essential idea would have been the same. Indeed, the Burrough Truth Function Evaluator as described in 1954 by Burks, Warren and Wright, [4], is realized in this way.

For reasons of simplicity of the design we were going as far as providing each column in the keyboard with its own logical unit. The logical units are built up conventionally by relays. Also the truth values are represented by zero or working voltage on the output of the five switches corresponding to the values 0 or 1 of the variables p, q, v, s, t. Our interest was not devoted to these points, but to an automatic connection of the logical units by a mechanized switching system which works according to Angstl's rule. This switching system contains an additional feature which shows whether the string of symbols on the keyboard is a formula or is not well-formed.

Details may be seen from the wiring diagram, Fig. 2. Application of the formula-controlled logical computer is most simple. One only has to push buttons on the keyboard which operates like a desk calculator keyboard and therewith to "write in" the formula. The truth value of the variables involved may be set on switches. A red or yellow light flashes on according to the value of the formula being "wrong" or "right" for these values. A special switch provides the test whether the string of symbols "written in" is a formula (blue light) or not well-formed.

The keyboard scheme may be seen from Fig. 3. The formula written-in is $KNpNq$ or $\bar{p} \wedge \bar{q}$ in common notation. The switch at the bottom of the keyboard is in position "Check for meaningfulness." After changing it to the other

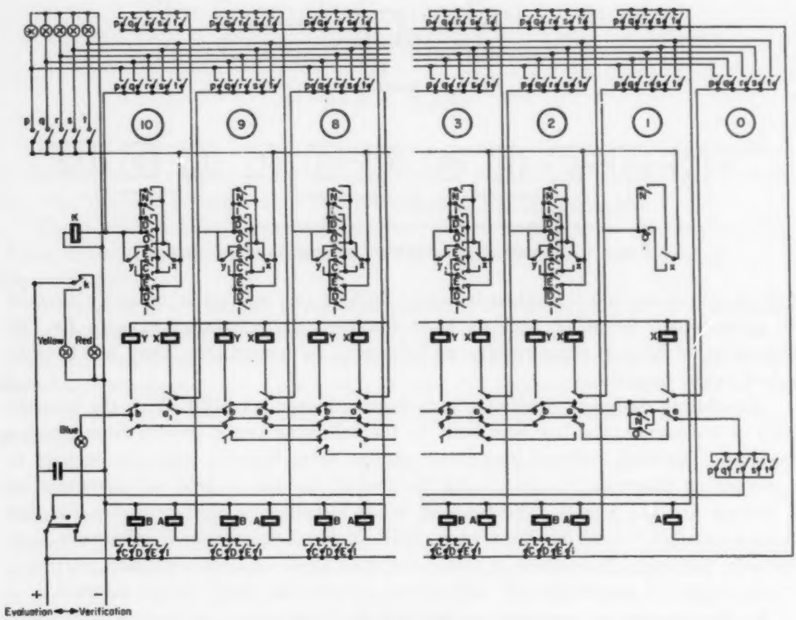

FIG. 2.—Wiring diagram.

Pushbuttons on the formula keyboard, columns 0–10; *O*, blank; *N*, negation; *C*, conjunction; *D*, disjunction; *E*, equivalence; *I*, implication; *p*; *q*, corresponding; *r*, variables; *s*; *t*.

Relays, columns 0–10; *X*, *Y*, logical circuits; *x*, *y*, corresponding contacts; *A*, *B*, transmission lines; *a*, *b*, corresponding contacts.

Relays and switches, left side: *p*, *q*, *r*, *s*, *t*, switches for truth values input.

Evaluation-Verification switch for evaluation or check; *K*, evaluation relay; *k*, corresponding contact.

FIG. 3.—Keyboard scheme.

side, the yellow result light will flash on, indicating the truth value of $KNpNq$ for p = wrong and q = wrong according to the left position of the switches for the variables p and q on the right side of the keyboard.

Johannes Gutenberg-Universität in Mainz,
Institut für Angewandte Mathematik,
Mainz, Saarstrasse 21, Germany

1. J. Lukasiewicz & A. Tarski, "Untersuchungen über den Aussagenkalkül," *Comptes Rendus des Séances* de la Société des Sciences et des Lettres de Varsovie, Cl. III, 23, p. 31–32, 1930; and Paul Rosenbloom, *The Elements of Mathematical Logic*, New York, 1950.
2. Private Communication, not published. Angstl reported on his results in a seminar on logistics held in 1950 at the University of Munich by Professor W. Britzelmayr.
3. W. Burkhart, "Electrical analysis of truth functions," Thesis.
4. Arthur W. Burks, Don Warren & Jesse B. Wright, "An analysis of a logical machine using parenthesis-free notation," *MTAC*, v. 8, 1954, p. 53–57.

Evaluation at Half Periods of Weierstrass' Elliptic Function with Rectangular Primitive Period-Parallelogram

By Chih-Bing Ling

The purpose of this paper is to evaluate the following Weierstrass' elliptic function at half periods [1],

$$e_1 = \wp(\omega_1), \qquad e_2 = \wp(\omega_2), \qquad e_3 = \wp(\omega_3), \tag{1}$$

where $2\omega_1$ and $2\omega_2$ are double periods of the function and ω_3 is defined by

$$\omega_1 + \omega_2 + \omega_3 = 0. \tag{2}$$

This paper tabulates only the values of the function whose primitive period-parallelogram is a rectangle with $2\omega_1 = 1$ and $2\omega_2 = ai$, where $a \geqq 1$.

The three functions in (1) form a set of distinct roots of the cubic [1]

$$x^3 - px - q = 0, \tag{3}$$

where

$$p = 15\sigma_4, \qquad q = 35\sigma_6, \tag{4}$$

and

$$\begin{aligned} \sigma_{2k} &= \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(2m\omega_1 + 2n\omega_2)^{2k}} \\ &= 2\sum_{m=1}^{\infty} \frac{1}{m^{2k}} + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{(m + nai)^{2k}}. \end{aligned} \tag{5}$$

The accent on the summation sign denotes the omission of simultaneous zero values of m and n from the double summation.

The cubic (3) indicates that

Received April 7, 1958; in revised form, August 7, 1959.

$$e_1 + e_2 + e_3 = 0. \tag{6}$$

Also, since e_1, e_2 and e_3 are distinct, the discriminant $(4p^3 - 27q^2)$ of the cubic does not vanish. As will be seen later, in the present case both σ_4 and σ_6 are real and $(4p^3 - 27q^2)$ is positive. This implies that all the roots of the cubic are real.

The evaluation of σ_4 and σ_6 is facilitated by using the known relation [2]

$$\cot x = \frac{1}{x} + \sum_{m=-\infty}^{\infty}{}' \left(\frac{1}{m\pi + x} - \frac{1}{m\pi}\right) \tag{7}$$

where the accent on the summation sign denotes the omission of the zero value of m from the summation. By repeated differentiation of Equation (7) and substitution of ix for x, it is found that

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \frac{1}{(m\pi + ix)^4} &= \frac{2}{3\sinh^2 x} + \frac{1}{\sinh^4 x} \\ \sum_{m=-\infty}^{\infty} \frac{1}{(m\pi + ix)^6} &= -\frac{2}{15\sinh^2 x} - \frac{1}{\sinh^4 x} - \frac{1}{\sinh^6 x}. \end{aligned} \tag{8}$$

Hence we have

$$\begin{aligned} \sigma_4 &= \frac{\pi^4}{45} + \frac{4\pi^4 K_1}{3\sinh^2 \pi a} \\ \sigma_6 &= \frac{2\pi^6}{945} - \frac{4\pi^6 K_2}{15\sinh^2 \pi a} \end{aligned} \tag{9}$$

where

$$\begin{aligned} K_1 &= \sinh^2 \pi a \sum_{n=1}^{\infty} \left(\frac{1}{\sinh^2 n\pi a} + \frac{3}{2\sinh^4 n\pi a}\right) \\ K_2 &= \sinh^2 \pi a \sum_{n=1}^{\infty} \left(\frac{1}{\sinh^2 n\pi a} + \frac{15}{2\sinh^4 n\pi a} + \frac{15}{2\sinh^6 n\pi a}\right). \end{aligned} \tag{10}$$

Consequently, we find

$$\frac{4p^3 - 27q^2}{16\pi^{12}} = \frac{5K_1 + 7K_2}{3\sinh^2 \pi a} + \frac{100K_1^2 - 147K_2^2}{\sinh^4 \pi a} + \frac{2000K_1^3}{\sinh^6 \pi a}. \tag{11}$$

With the aid of known tables [3, 4], values of K_1, K_2, and then σ_4, σ_6 and $(4p^3 - 27q^2)^{\frac{1}{2}}$ are computed to 16D for $a = 1(0.25)2(1)6$ and ∞ as shown in Table 1.

The subsequent evaluation of e_1, e_2, and e_3 requires the solution of the cubic (3). It appears that one of the roots, e_1, can be easily evaluated to 16D as shown in Table 2 by using Newton's method or otherwise, but difficulty arises in evaluating the other two roots for in most cases they are almost equal. However, they can be separated by forming a new cubic

$$x^3 + p'x - q' = 0 \tag{12}$$

whose roots are the differences of the roots of the cubic (3). Let $(e_1 - e_2)$, $(e_2 - e_3)$ and $(e_3 - e_1)$ be the roots of the new cubic. We have

$$\begin{aligned} p' &= (e_1 - e_2)(e_2 - e_3) + (e_2 - e_3)(e_3 - e_1) + (e_3 - e_1)(e_1 - e_2) = -3p, \\ q'^2 &= (e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = 4p^3 - 27q^2. \end{aligned} \tag{13}$$

TABLE 1

a	K_1	K_2	ω_1	ω_3	$(4p^3 - 27q^2)^{\frac{1}{2}}$
1	1.01311,06293,09539	1.05851,88947,76779	3.15121,20021,53898	0	6.49955,04200,35218 × 10^2
1.25	1.00271,90816,67748	1.01206,13101,78251	2.36702,93923,35617	1.63147,62559,94511	3.01664,67968,60964 × 10^2
1.5	1.00056,49682,73190	1.00250,28510,82885	2.20660,15468,91272	1.95170,97194,76020	1.38049,25551,66708 × 10^2
1.75	1.00011,74335,66488	1.00052,00996,05499	2.17336,30560,32070	2.01747,33403,07279	6,29902,26479,45028 × 10
2	1.00002,44115,30272	1.00010,81097,89987	2.16645,82514,80805	2.03110,95062,61006	2.87242,26104,19235 × 10
3	1.00000,00455,86885	1.00000,02018,84784	2.16464,98507,19257	2.03467,94456,07301	1.24133,82088,92023
4	1.00000,00000,85131	1.00000,00003,77008	2.16464,64737,40389	2.03468,61114,97443	5.36430,92081,07968 × 10^{-2}
5	1.00000,00000,00159	1.00000,00000,00704	2.16464,64674,34075	2.03468,61239,45609	2.31812,81969,45459 × 10^{-3}
6	1.00000,00000,00000	1.00000,00000,00001	2.16464,64674,22298	2.03468,61239,68855	1.00175,40242,77741 × 10^{-4}
∞	1.00000,00000,00000	1.00000,00000,00000	2.16464,64674,22276	2.03468,61239,68898	0

TABLE 2

a	e_1	$-(e_3 - e_2)$	$-e_3$	$-e_2$
1	6.87518,58180,20373	6.87518,58180,20373	6.87518,58180,20373	0
1.25	6.64106,26950,12943	3.11618,71546,58608	4.87862,49248,35775	1.76243,77701,77167
1.5	6.59248,08531,44224	1.41904,24293,36329	4.00576,16412,40277	2.58671,92119,03947
1.75	6.58238,54370,71645	6.46830,14416,89006 × 10^{-1}	3.61460,77906,20273	2.96777,76464,51372
2	6.58028,69683,44880	2.94898,84967,54931 × 10^{-1}	3.43759,29090,10186	3.14269,40593,34693
3	6.57973,72957,91816	1.27435,57352,40785 × 10^{-2}	3.29624,04265,72112	3.28349,68692,19704
4	6.57973,62693,13382	5.50699,03149,79195 × 10^{-4}	3.29014,34841,72440	3.28959,27851,40942
5	6.57973,62673,96492	2.37978,62933,93412 × 10^{-5}	3.28988,00326,29713	3.28985,62347,66779
6	6.57973,62673,92912	1.02839,89036,79391 × 10^{-6}	3.28986,86478,95908	3.28986,76194,97004
∞	6.57973,62673,92906	0	3.28986,81336,96453	3.28986,81336,96453

Consequently, by taking a positive sign for q', the new cubic is in the form

$$x^3 - 3px - (4p^3 - 27q^2)^{\frac{1}{2}} = 0. \tag{14}$$

From this cubic, values of $(e_2 - e_3)$ and then e_2 and e_3 are computed to $16D$ as shown in Table 2.

It is mentioned that the values of the function, for $0 \leqq a < 1$ or in general for ω_2/ω_1 purely imaginary, can be computed from the tabulated values with the aid of the following relation [1]

$$\wp(\lambda z \mid \lambda\omega_1, \lambda\omega_2) = \lambda^{-2}\wp(z \mid \omega_1, \omega_2) \tag{15}$$

where λ is a constant, real or complex.

The writer wishes to express his thanks to Mr. C. P. Tsai for his assistance in performing the numerical computations. The writer also is deeply grateful to Professor C. W. Nelson of the University of California, Berkeley, for checking the manuscript and verifying all the numerical values in Tables 1 and 2 by independent calculations. Thanks are also due to the referee of the paper, who suggests a different method of computation [5] without solving the cubic equation.

Institute of Mathematics
Academia Sinica
Taiwan, China

1. E. T. Copson, *Theory of Functions of a Complex Variable*, Oxford University Press, New York, 1935, p. 359–362.
2. E. P. Adams & R. L. Hippisley, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Smithsonian Miscellaneous Collection 74, Smithsonian Institution, Washington, D. C., 1939, p. 129.
3. J. W. L. Glaisher, "Tables of $1 \pm 2^{-n} + 3^{-n} \pm 4^{-n} +$ etc., and $1 + 3^{-n} + 5^{-n} + 7^{-n} +$ etc., to 32 places of decimals," *Quart. Jn. Pure and Appl. Math.*, v. 45, 1914, p. 141–158.
4. British Association for the Advancement of Science, Mathematical Tables Volume 1, *Circular and Hyperbolic Functions, etc.*, University Press, Cambridge, 1946, p. 24–29.
5. A. Erdelyi, et al., *Higher Transcendental Functions*, v. 2, McGraw-Hill Book Co., New York, 1953, p. 361.

A Note on the Nonexistence of Certain Projective Planes of Order Nine

By Raymond B. Killgrove

1. Introduction. Every finite projective plane may be coordinatized in at least one way [1]. In this process some line is chosen to be the line at infinity, and the points not on this line are represented by an ordered pair of elements. The elements x and y for any point (x, y) on a given line of the plane satisfy the equation $y = x \cdot m \circ b$, where m and b are specific elements for the given line. This ternary operation on x, m, and b includes an additive loop in a special case.

A sequence of SWAC computer routines has been written to search for all planes having a specific additive loop in an appropriate ternary ring. Using these routines, a complete search had been made previously using the elementary Abelian group for the additive loop [2]. Now a complete search has been made using the

Received August 19, 1959. The work of Mr. Killgrove and the preparation of this paper were supported in part by the Office of Naval Research.

cyclic group for the additive loop. No planes were found. This entire note parallels the paper [2].

2. The lines consistent with the cyclic additive pencil. It is sufficient to find the affine planes of order nine where the lines parallel to the line $y = x$ are determined by the cyclic group. In particular these lines are of the form $y = x + c$ where y may be computed for each x and a given c by addition modulo nine.

We represent any possible line for a geometry as a permutation on nine marks in the following way: a number a and its image b under the permutation define a point (a, b) of the line. All of these lines can interesect in at most one point with the lines of the cyclic groups. We can also restrict ourselves to the pencil of lines going through the origin (0, 0). Only 225 lines were found which satisfy these criteria.

By applying the permutations defined by the additive loop, the full list of 2,025 lines consistent with the cyclic group may be found. Having the lines $x = c$, $y = c$, in addition to $y = x + c$, one needs to find sets of 63 lines from the 2,025 which satisfy the affine plane axioms.

To save time one considers the automorphisms which preserve the 27 known lines, and in particular the subgroup of automorphisms which fix (0, 0). The cyclic group of order nine has such an automorphic subgroup of order 6. When this subgroup is applied to the 31 lines of the 225 containing the point (1, 2), one obtains 25 conjugate classes of lines. The conjugate classes are placed in some preference order. In this case the order chosen was 4, 16, 8, 9, 10, 11, 13, 15, 18, 19, 20, 23, 24, 25, 2, 3, 6, 7, 17, 21, 12, 14, 1, 22, 5. All but the last five classes mentioned in this order have six members. Conjugate classes 12 and 14 have three members, 1 and 22 have two members, and 5 has only one member.

3. Construction of pencils through (0, 0). Using a card sorter one finds the lines of the reduced set consistent with each line through (1, 2). Conjugate classes of lines preceding the current sorting line in preference order are removed also. This set of sorted cards is given the computer, which in turn does further sorting and the printing out of complete Latin squares. On the average each conjugate class yielded eight Latin squares. These Latin squares represent possible pencils of lines through (0, 0).

Next we use a set of 225 transformed lines which are obtained by applying one of the non-identity permutations to the original 225 lines. The computer then lists the transformed lines consistent with a given Latin square. In the previous elementary Abelian group case [2], whenever the list of transformed lines associated with some Latin square could themselves form a second Latin square, one usually finds a geometry. In this cyclic group case a second consistent Latin square was never found. Hence the search ended here, as there are no possible planes in this situation.

Department of Mathematics
University of California
Los Angeles, California

1. Marshall Hall, Jr., "Projective planes," *Trans. Amer. Math. Soc.*, v. 54, 1943, p. 229–277.

2. Marshall Hall, Jr., J. Dean Swift & Raymond Killgrove, "On projective planes of order nine," *MTAC*, v. 13, 1959, p. 233–246.

A Note on Rational Approximation

By Robert W. Floyd

It is suggested by plausible reasoning and confirmed by experience that the error of an nth degree polynomial approximation, in the Chebyshev sense of least maximum error, to an analytic function, is roughly a multiple of the $n + 1$st Chebyshev polynomial, $T_{n+1}(x)$, on the interval of approximation. Therefore if the nth degree polynomial $f^*(x)$ is equal to the function, $f(x)$, on the roots of $T_{n+1}(x)$, we expect that $f^*(x)$ will be a satisfactory approach to a Chebyshev approximation of $f(x)$.

Because $f(x)$ is analytic, it may be represented with negligible error in the interval of approximation by a polynomial $p(x)$ of sufficiently high degree; e.g., a truncated Taylor's or Maclaurin's series. Applying the division algorithm for polynomials,

$$\begin{aligned} p(x) &= q_0(x)\cdot T_{n+1}(x) + r_0(x) \\ T_{n+1}(x) &= q_1(x)\cdot r_0(x) + r_1(x) \\ r_0(x) &= q_2(x)\cdot r_1(x) + r_2(x) \\ r_1(x) &= q_3(x)\cdot r_2(x) + r_3(x), \text{ etc.,} \end{aligned}$$

where the degrees of the r_i form a strictly decreasing sequence. From these equations we may write $r_i(x) = a_i(x)\cdot p(x) + b_i(x)\cdot T_{n+1}(x)$, where a_i and b_i are defined recursively by

$$\begin{aligned} a_i &= a_{i-2} - q_i\cdot a_{i-1}, \quad & a_{-1} &= 0, \quad & a_{-2} &= 1 \\ b_i &= b_{i-2} - q_i\cdot b_{i-1}, \quad & b_{-1} &= 1, \quad & b_{-2} &= 0. \end{aligned}$$

It may be proven that the sum of the degrees of $a_i(x)$ and $r_i(x)$ is at most n. The first set of equations may be written $p(x) = [r_i(x)/a_i(x)] - [b_i(x)/a_i(x)]\cdot T_{n+1}(x)$, so that $r_i(x)/a_i(x)$ is a rational approximation to $p(x)$, exact wherever $T_{n+1}(x)$ vanishes. Since $T_{n+1}(x) \leqq 1$ in the interval of approximation, $b_i(x)/a_i(x)$ provides a bound for the error of the approximation. If $b_i(x)/a_i(x)$ is nearly constant on the interval of approximation, the error oscillates between $n + 2$ extrema of nearly equal magnitude, and the method of approximation is justified, for Chebyshev approximation is characterized by an error which oscillates at least $n + 1$ times between positive and negative extrema of equal magnitude. For the particular case $i = 0$, $a_i = 1$, and $r_0(x)$ is a polynomial approximation to $f(x)$ of degree at most n.

For example; $f(x) = e^x = 1 + x + (x^2/2!) + (x^3/3!) + \cdots$;

$$\begin{aligned} p(x) = 1 + x + .5x^2 + .1666\,6667x^3 + .0416\,6667x^4 + .0083\,3333x^5 \\ + .0013\,8889x^6 + .0001\,9841x^7 + .0000\,2480x^8 + .0000\,0276x^9. \end{aligned}$$

For $-1 \leqq x \leqq 1$, $|\,p(x) - f(x)\,| \leqq 3.0 \times 10^{-7}$. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$. Then $q_0 = (317.5625 + 38.75x + 4.3125x^2) \times 10^{-8}$;

Received July 6, 1959.

$$r_0 = 1 + 1.0000\,2223x + .5000\,0271x^2 + .1664\,8913x^3 + .04164497x^4 + .00868659x^5 + .0014\,3229x^6;$$

$$|p(x) - r_0| = |q_0|\cdot|T_7(x)| \leqq 3.61 \times 10^{-6} \quad (-1 \leqq x \leqq 1).$$

Therefore $|f(x) - r_0| \leqq 3.91 \times 10^{-6}$ $(-1 \leqq x \leqq 1)$. Dividing $T_7(x)$ by r_0,

$$q_1 = -270{,}998.81 + 44{,}683.688x.$$

$$r_1 = 270{,}998.81 + 226{,}314.15x + 90{,}815{,}458x^2 + 22{,}832.391x^3 + 3{,}846.3890x^4 + 381.2048x^5.$$

$$a_0 = 1;\ b_0 = -q_0$$

$$a_1 = -q_1;\ b_1 = 1 + q_1q_0$$

Therefore

$$p(x) = \frac{r_1}{a_1} - \frac{b_1}{a_1}T_7 = -\frac{r_1}{q_1} + \frac{1 + q_1q_0}{q_1}T_7(x).$$

The second term on the right is

$$\frac{.1394\,0940 + .0368\,86598x + .0056\,281054x^2 + .0019\,269840x^3}{-270{,}998.81 + 44{,}683.688x}T_7(x)$$

whose absolute value is bounded by 8.121×10^{-7} for $-1 \leqq x \leqq 1$. Thus e^x may be approximated on this interval by

$$-\frac{r_1}{q_1} = \frac{1 + .8351\,1123x + .3351\,1386x^2 + .0842\,5274x^3 + .0141\,9338x^4 + .0014\,0667x^5}{1 - .1648\,8518x},$$

where the error is bounded by $\pm(3 \times 10^{-7} + 8.1 \times 10^{-7}) = \pm 1.1 \times 10^{-6}$.

Armour Research Foundation
Illinois Institute of Technology
Chicago 16, Illinois

1. C. Hastings, *Approximations for Digital Computers*, Princeton, 1955, p. 47–64.
2. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill Book Co., New York, 1956, p. 389–395.
3. NBS Applied Mathematics Series, 9, *Tables of Chebyshev Polynomials $S_n(x)$ and $C_n(x)$*, U. S. Govt. Printing Office, Washington, D. C., 1952, p. 16–18.

The Complete Factorization of $2^{132} + 1$

By K. R. Isemanger

The integer $2^{132} + 1$ is divisible by $2^{44} + 1 = 17\cdot353\cdot2931542417$ and the quotient, $2^{88} - 2^{44} + 1$, is divisible by $241\cdot7393$. There remains the formidable problem of factoring the resultant quotient N, where N is the integer

$$1\ 73700\ 82040\ 22350\ 83057.$$

Received July 3, 1959; in revised form, September 17, 1959.

By a method of exclusion, using small prime moduli, the following binary representations of N were obtained:

$$N = 88073\ 81316^2 + 98046\ 34351^2$$

$$N = 82086\ 25547^2 + 2^2 \cdot 3^2 \cdot 11^2 \cdot 37^2 \cdot 4243^2 \cdot 7 \cdot 17 \cdot 19 \cdot 73$$

$$N = 1\ 28399\ 72408^2 + 7^2 \cdot 11^2 \cdot 13^2 \cdot 3137^2 \cdot 3 \cdot 19 \cdot 79 \cdot 199$$

$$N = 23334\ 17999^2 + 2^4 \cdot 13^2 \cdot 17^2 \cdot 2707^2 \cdot 3 \cdot 7 \cdot 79 \cdot 89 \cdot 199$$

$$N = 1\ 06383\ 10009^2 + 2^{10} \cdot 17^2 \cdot 3853^2 \cdot 3 \cdot 107 \cdot 167 \cdot 257$$

$$N = 1\ 05791\ 40907^2 + 2^2 \cdot 3^2 \cdot 13^2 \cdot 3169^2 \cdot 17 \cdot 23 \cdot 29 \cdot 89 \cdot 167$$

$$N = 1\ 58709\ 07595^2 - 2^2 \cdot 3^2 \cdot 17^2 \cdot 37^2 \cdot 3061^2 \cdot 7 \cdot 13 \cdot 29$$

$$N = 1\ 47403\ 24637^2 - 2^2 \cdot 7^2 \cdot 23^2 \cdot 73^2 \cdot 2803^2 \cdot 3 \cdot 7 \cdot 239$$

$$N = 1\ 51299\ 86183^2 - 2^9 \cdot 7^2 \cdot 3259^2 \cdot 3 \cdot 37 \cdot 73 \cdot 107 \cdot 239$$

$$N = 1\ 49963\ 46199^2 - 2^5 \cdot 3^4 \cdot 11^2 \cdot 17^2 \cdot 2711^2 \cdot 13 \cdot 23 \cdot 257$$

If these ten relations are written as congruences in the form

$$x^2 \equiv Dy^2 (\text{mod } N)$$

and then are multiplied together, there results the congruence

$$A^2 \equiv B^2 (\text{mod } N)$$

where

$$A = 3030\ 76720\ 24193\ 56872$$

$$B = 85638\ 82032\ 43137\ 98848.$$

The greatest common divisor of $A + B$ and N was found to be 9 86182 73953, which yields the factorization

$$N = 17613\ 45169 \cdot 9\ 86182\ 73953.$$

The primality of both factors was verified on the SILLIAC computer at the University of Sydney.

This result supplements information previously published by R. M. Robinson [1].

19 Snape Street
Kingsford, Sydney
New South Wales, Australia

1. Raphael M. Robinson, "Some factorizations of numbers of the form $2^n \pm 1$," *MTAC*, v. 11, 1957, p. 265–268.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

1[B, C, K, S].—A. H. WAPSTRA, G. J. NIJGH & R. VAN LIESHOUT, *Nuclear Spectroscopy Tables*, North Holland Publishing Company, Amsterdam, 1959, viii + 136 p., 27 cm. Price $8.90.

This book consists of a well-selected collection of tables and graphs of interest to nuclear physicists. A slightly condensed list of contents follows:

Chapter I. *Mathematical Data.*

Common logarithms. Powers of 10 and 2, for exponents 1.00(0.01)9.99, 4D. Cube roots of integers 1(1)999, 5D. Brief discussion of least-squares method, and a graph for checking the consistency of least-squares computations. Table of Gaussian distribution and its integral for arguments 0.00(0.02)3.48, 4D.

Chapter II. *Tables of Atomic Constants.*

Chapter III. *Elements and Isotopes.*

Names, atomic numbers, and symbols of elements. Atomic weights and abundances of isotopes.

Chapter IV. *Heavy Particles.*

Range energy curves for protons, deuterons, and alpha particles. Straggling of protons in air (graph). Magnetic rigidity of protons, deuterons, and alpha particles (table). Half-lives for alpha disintegration and spontaneous fission of heavy elements (graphs).

Chapter V. *Electrons.*

Range energy curves for beta-particles. Saturation backscattering coefficient (graph). Magnetic rigidity of electrons (table). Discussion of shapes of continuous beta-spectra. Reduced Fermi function $f(z, p)$ and beta-decay functions L and M for negatrons and positrons (tables). Screening correction for negatrons and positrons (graphs). Discussion of beta-decay transition probabilities and their values. Ratios of electron capture in different electron shells. Computation of log ft values (nomogram and graphs). K capture/positron emission ratios (table and graphs).

Chapter VI. *Gamma Rays.*

Half-thickness of some substances for absorption of gamma rays vs energy (graph). Photon absorption cross-sections vs energy (table). Energy of Compton scattered gamma rays (discussion and graph). Gamma-decay half-lives (discussion, nomogram, and graphs).

Chapter VII. *X-rays and Auger Electrons.*

Electron binding energies in different shells for all elements (table). Relative intensities of K X-ray components and of K-Auger electrons (table).

Chapter VIII. *Angular Distributions and Correlations.*

Brief discussion of angular distributions and correlations involving photons, alpha particles, beta rays and K conversion electrons. Tables of $F_\nu(L, L', j_i, j)$ coefficients.

Chapter IX. *Nuclear Models.*

Discussion of nuclear mass formula, nuclear shell model, collective model, magnetic moments, quadrupole moments, and gamma- and beta-decay

probabilities. Graphs of Nilsson level scheme of single particle orbits in spheroidal potential. Table of measured ground state spins of odd-A and odd-odd nuclei. Tables of Clebsch-Gordan coefficients.

Chapter X. *Calibration Standards.*

Tables of standard gamma and electron lines and of standard alpha rays. Gamma-ray absorption coefficient in NaI crystals. Table of standard nuclides for calibration of gamma-ray spectrometer.

The tables and graphs have been presented so as to be easily read, and the quality of the printing is good. Much of the material is used frequently by nuclear physicists but is widely scattered in the literature. Thus, this book should prove very helpful to people in the field of nuclear physics, and this reviewer recommends it highly.

MICHEL A. MELKANOFF

University of California
Los Angeles, California

2[D, L].—L. K. FREVEL, J. W. TURLEY & D. R. PETERSEN, *Seven-Place Table of Iterated Sine*, The Dow Chemical Company, Midland, Michigan, 1959. Deposited in UMT File.

Following a detailed description of the method of computation employed, the authors give a 7D table of the nth iterated sine function of x for $n = 0(.05)10$, and $x = k(\pi/20)$, where $k = 1(1)10$. It is stated that the computations were per-performed on a Datatron 204, and the results are considered correct to within $5 \cdot 10^{-7}$.

J. W. W.

3[G].—K. M. HOWELL, *Revised Tables of 6j-Symbols*, U. Southampton Math. Dept., Research Report 59-1, 1959, xvi + 181 p., 33 cm.

The Wigner $6j$-symbol has been defined by Wigner in general in connection with the reduction of the triple Kronecker product of any simply reducible group. In these tables this group is taken to be either the three-dimensional rotation group or the two-dimensional unitary group. The symbols are denoted by

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{Bmatrix}$$

where the quantities $j_1, \cdots, k_3$ are integers or half-integers. If we let

$$J_0 = j_1 + j_2 + j_3, \qquad J_1 = j_1 + k_2 + k_3, \qquad J_2 = j_2 + k_1 + k_3$$

$$J_3 = j_3 + k_1 + k_2$$

$$K_1 = j_1 + j_2 + k_1 + k_2 \qquad K_2 = j_1 + j_3 + k_1 + k_3$$

$$K_3 = j_2 + j_3 + k_2 + k_3,$$

then the explicit expression for the $6j$-symbol is

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{Bmatrix} = \{\prod_{r,s} (K_r - J_s)!/\prod_s (J_s + 1)!\}^{\frac{1}{2}}$$

$$\cdot \sum_t (-1)^t (t+1)!/\{\prod_r (K_r - t)! \prod_s (t - J_s)!\}.$$

These symbols are invariant under the 144 products of separate permutations of the K_r alone and the separate permutations of the J_s alone. The symmetries of the 6j-symbols are made use of in the organization of the tables. In the present (revised) edition of the tables a wider group of symmetries is exploited than in the earlier edition.

In the tables the 6j-symbols are classified in terms of a set of six ordered parameters, and the tables are arranged in descending "speedometer" order of these parameters. Rules for determining the values of these parameters from given $j_1, \cdots, k_3$ are included.

The square of the value of the 6j-symbol is printed in the table, together with its correct sign. In addition the entries are written as rational fractions in terms of their prime factors. Thus the fraction $-4/5\sqrt{21}$ is written as $-2\cdot2\cdot2\cdot2/3\cdot5\cdot5\cdot7$. A composite member, all of whose factors are greater than 103 is printed as if it were a prime.

The tables were duplicated from stencils cut directly from the output tape of a Ferranti Pegasus Computer. The program used in the computer was essentially the same as that used for the earlier version of the tables. It is stated that, "It seems reasonably sure that these tables are as reliable as, and more comprehensive than, the first set of tables of the 6j-symbol."

A. H. T.

4[H, M].—H. J. Gawlik, *Zeros of Legendre Polynomials of orders 2–64 and weight coefficients of Gauss quadrature formulae*, A. R. D. E. (Armament Research and Development Establishment) Memorandum (B) 77/58, Fort Halstead, Kent, December 1958, 25 p.

The Gauss quadrature formula is $\int_{-1}^{1} f(x)\,dx = \sum_{r=1}^{n} A_r f(a_r)$ with A_r, a_r chosen so that the equality is valid whenever $f(x)$ is any polynomial of degree $2n - 1$ or less. The quantities a_r are the zeros of the Legendre polynomial $P_n(x)$, while

$$A_r = \frac{1}{P_n'(a_r)} \int_{-1}^{1} \frac{P_r(x)}{x - a_r}\,dx.$$

The memorandum here reviewed gives 20-decimal values of a_r and A_r for all zeros a_r of each polynomial $P_n(x)$, for $n = 2(1)64$, and is by far the most extensive and accurate of such tables available.

The most extensive of the previously published tables is that of Lowan, Davids, and Levinson [1], which gives a similar table to 15 decimals for $n = 2(1)16$. This table has been reproduced several times, in whole or in part [2]. It has been supplemented by a table by Davis and Rabinowitz [3], which gives 20-decimal values for n = 2, 4, 8, 16, 20, 24, 32, 40, 48. Discrepancies between Davis & Rabinowitz and Gawlik amount to only a unit in the 20th decimal in several places, with the probability that Gawlik is the more accurate; both tables would thus appear to be reliable.

J. C. P. Miller

The University Mathematical Laboratory
Cambridge, England

1. Amer. Math. Soc., *Bull.*, v. 48, 1942, p. 739–743; v. 49, 1943, p. 939 (errata).
2. See, for example, *NBS* Applied Mathematics Series, No. 37, *Tables of Functions and of Zeros of Functions*, 1954.
3. NBS, *Jn. of Research*, v. 56, 1956, p. 35–37.

5[I].—A. N. LOWAN, *The Operator Approach to Problems of Stability and Convergence of Solutions of Difference Equations and the Convergence of Various Iteration Procedures*, Scripta Mathematica, New York, 1957, x + 104 p., Office of Technical Services, Department of Commerce, Washington 25, D. C. 26 cm. Price $3.00.

In the numerical solution of linear second-order differential equations by difference methods, one has to solve

$$A_k u_{k+1} + B_k u_k + C_k u_{k-1} = D_k \tag{1}$$

where A_k, B_k, C_k, D_k are sparse matrices with regular structure, the u's are vectors, and the integer subscripts refer to time-steps, iteration cycles, etc. In many important cases A_k, B_k, C_k are independent of k and closely related. Stability and mesh-convergence of "stepping-ahead" solutions (parabolic, hyperbolic equations) and convergence of iterative solutions (elliptic equations) can be discussed in terms of the operators $A_k^{-1}B_k$ and $A_k^{-1}C_k$. In the von Neumann technique, as formalized and extended by the reviewer, one essentially takes the Fourier transform (in the extended sense) of (1), thus introducing immediately the eigenvectors and eigenvalues of these operators. This is equivalent to a change of coordinates in vector space under which A_k, B_k, C_k take very simple forms. Various authors, however, have retained the original coordinates and worked directly with (1); prominent among these are S. Frankel, D. Rutherford, A. Mitchell, P. Lax, R. Richtmyer, J. Douglas, J. Todd, and (unpublished) C. Leith. Professor Lowan here gives a detailed, connected account of this second method which he calls the "operator approach". He covers the usual parabolic, elliptic, and hyperbolic partial differential equations, homogeneous and non-homogeneous, with some attention to equations with variable coefficients. He discusses stability and mesh-convergence for parabolic and hyperbolic equations, and convergence of the various iterative schemes for elliptic equations. Original contributions, besides the organization of the material, include a discussion of iteration schemes for solving "implicit" difference approximants to parabolic and hyperbolic equations and a novel "second-order" Richardson scheme for elliptic difference equations. In addition, Professor Lowan has written down a number of "folk theorems", rather widely known but unpublished. There are eight "Sections" and six "Appendices". Most of the typographical errors have been detected by the author and listed on the "Errata" sheet. However, the figures on page 5 and at the top of page 33 should be corrected; in the second line from the bottom of page 44, read $(-1)^{h+1}\sqrt{2}\sin rh\pi/M$; page 26, line 5 and page 47, line 18 should show the respective qualifications "$r > 1$" and "$r \leqq \frac{1}{2}$". Moreover, the reader should be aware that the estimates of truncation error in difference solutions given during the discussion of mesh-convergence require that various functions be "sufficiently continuous." Those of us interested in the numerical solution of partial differential equations are indebted to Professor Lowan for a very worthwhile addition to the literature of this field.

MORTON A. HYMAN

IBM Research Laboratory
Yorktown Heights, New York

6[K].—CHARLES W. DUNNETT, "Tables of the bivariate normal distribution with correlation $1/\sqrt{2}$," 1958, 28 cm. Deposited in UMT File.

The bivariate normal probability distribution,

$$L(h, k; r) = \frac{1}{2\pi\sqrt{1 - r^2}} \int_h^\infty \int_k^\infty \exp\left[-\tfrac{1}{2}\left(\frac{x^2 - 2rxy + y^2}{1 - r^2}\right)\right] dx\, dy,$$

has been tabulated by Karl Pearson [1] for $r = -1.0(.05)1.0$. The present tables were prepared to avoid the necessity of interpolating in Pearson's tables when $r = \pm 1/\sqrt{2}$. This case arises in certain double sampling procedures in which probability statements concerning X and $X + Y$ jointly are required, where X and Y are independent normal chance variables with the same variance.

The tables were computed on a Royal McBee LGP-30 electronic computer, by numerical quadrature, using the relation

$$L(h, k; r) = \int_h^\infty \left[1 - F\left(\frac{k - rx}{1 - r^2}\right)\right] f(x)\, dx$$

where $f(x) = (1/\sqrt{2\pi}) \exp(-x^2/2)$ and $F(x) = \int_{-\infty}^x f(x)\, dx$. The function is tabulated for $r = 1/\sqrt{2}$ in Table I and for $r = -1/\sqrt{2}$ in Table II for positive values of its arguments, h varying in steps of 0.1, and k varying in steps of $0.1\sqrt{2}$. All entries are given to six decimals and should be correct to this number of places.

The function can be determined for negative values of its arguments by using the relationships

$$L(-h, k; r) = L(-\infty, k; r) - L(h, k; -r)$$
$$L(h, -k; r) = L(h, -\infty; r) - L(h, k; -r)$$
$$L(-h, -k; r) = L(h, k; r) + 1 - L(h, -\infty; r) - L(-\infty, k; r)$$

where $L(h, -\infty; r) = L(-\infty, h; r) = 1 - F(h)$, which is the right-hand tail area of the univariate normal distribution. In order to avoid the necessity of consulting tables of $F(h)$, these values are included in the table.

Tables III, IV and V were computed from Tables I and II using these relationships. The error in the entries in Tables III and IV should be no greater than a unit in the sixth decimal place. The error in the entries in Table V should be no greater than two units in the sixth decimal place. All tables have been deposited in the Unpublished Mathematical Tables repository.

AUTHOR'S ABSTRACT

1. KARL PEARSON, *Tables for Statisticians and Biometricians*, Part II, Cambridge University Press, 1931.

7[L].—E. A. CHISTOVA, *Tablitsy funktsiĭ Besselia ot deĭstvitel'nogo argumenta i integralov ot nikh (Tables of Bessel functions of real argument and of integrals involving them)*, Izdatel'stvo Akademiĭ Nauk SSSR, Moscow, 1958, 524 p. + loose card, 28 cm. Price 45 rubles.

This important volume in the now familiar series of Mathematical Tables of the Computational Center of the Academy of Sciences was initiated by V. A. Ditkin.

The main table occupies pages 23–522 and gives values for

$$x = 0(.001)15(.01)100$$

of the eight functions

$$J_n(x), \qquad Ji_n(x) = \int_x^\infty \frac{J_n(u)}{u}\,du,$$

$$Y_n(x), \qquad Yi_n(x) = \int_x^\infty \frac{Y_n(u)}{u}\,du,$$

where $n = 0$ and 1. The values are to 7D or, near singularities at the origin, 7S Auxiliary functions (detailed below) are provided for $x = 0(.001).150$. Over the range $x = 1.350(.001)15$ there are no differences, and linear interpolation provides results correct within two units of the last place. Second differences are required for some functions in parts of the ranges $x = .150(.001)1.350$ and $x = 15(.01)100$; the quantities printed (when greater than about 16) are sums of two consecutive second differences, for use in Bessel's formula.

The values of $J_0(x)$ and $J_1(x)$ are rounded to 7D from the well-known Harvard tables, and are included for convenience; the values of the other six quantities result from original computations, and may be checked only partially against previous, less extensive, tables which are listed in a bibliography. The integrals $Ji_0(x)$ and $Ji_1(x)$ were computed by Simpson's rule on an electronic computer and other machines. The functions $Y_0(x)$, $Y_1(x)$, $Yi_0(x)$, and $Yi_1(x)$ were evaluated on the electronic computer BESM, using Taylor series and asymptotic expansions. All values were checked by differencing. By means of formulas given on pages 11–12, the integrals of $J_0(u)$, $J_1(u)$, $Y_0(u)$, and $Y_1(u)$, *not* divided by u, may be simply expressed in terms of the tabulated functions.

The nine auxiliary functions given on pages 17–19 are all tabulated for $x = 0.(.001).150$ to 7D without differences. The functions are:

$$Li_0(x) = Ji_0(x) + \ln \tfrac{1}{2}x$$

$$C_0(x) = Y_0(x) - (2/\pi)J_0(x)\ln x \qquad E_0(x) = (2/\pi)\{Ji_0(x) + \ln x\}$$

$$C_1(x) = x\{Y_1(x) - (2/\pi)J_1(x)\ln x\} \qquad E_1(x) = (2/\pi)\{Ji_1(x) - 1\}$$

$$D_0(x) = (2/\pi)J_0(x) \qquad F_0(x) = Yi_0(x) - (2/\pi)\ln x\{Ji_0(x) + \tfrac{1}{4}\ln x\}$$

$$D_1(x) = (2/\pi)J_1(x) \qquad F_1(x) = x\{Yi_1(x) + (2/\pi)\ln x[1 - Ji_1(x)]\}$$

It is stated that the errors do not exceed 0.6 final unit, except that they may attain one final unit in the case of the auxiliary functions $C_0(x)$, $C_1(x)$, $F_0(x)$, and $F_1(x)$.

A table of the Bessel coefficient $\frac{1}{4}t(1 - t)$ for $t = 0(.001)1$ to 5D without differences is given on page 523 and also on a loose card.

A. F.

8[L].—Otto Emersleben, "Werte einer Zetafunktion zweiter Ordnung mit Argument $s = 2$," Bearbeitet in der Abteilung Angewandte Mathematik der Universität Greifswald, Greifswald, 1956.

This paper contains a tabulation of the function

$$Z(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty}{}' \frac{e^{2\pi i(kx+ly)}}{k^2 + l^2}$$

where the prime denotes the fact that the term with $k = l = 0$ is omitted.

The table is given for values x and y in the range

$$\tfrac{1}{2} \geqq x \geqq y \geqq 0,$$

where x and $y = 0(0.01)0.5$. The entries are given to six and sometimes seven decimal places, and are said to be accurate to at least two units in the last decimal place.

In the calculation of this table, use was made of the seven-place tables of the exponential integrals published in 1954 by the U.S.S.R. Academy of Science, Institute of Exact Mechanics and Computation.

A. H. T.

9[L].—HERMAN H. LOWELL, *Tables of the Bessel-Kelvin Functions Ber, Bei, Ker, Kei, and their Derivatives for the Argument Range 0(0.01)107.50*, Technical Report R-32, National Aeronautics and Space Administration, Washington, D. C., 1959, 300 p., 26 cm. 2.25

These tables provide an elaborate and attractively arranged compilation of decimal values of the Bessel-Kelvin functions (frequently referred to simply as the Kelvin functions) of the first and second kinds of order zero, together with their first derivatives. Approximations to ber and bei and their first derivatives appear in floating-point form to generally 13 or 14 significant figures. On the other hand, the number of significant figures given for ker and kei and their first derivatives vary from 9 to 13, according to a pattern explained in the detailed introduction, which also describes the construction of these tables and the checks applied to the tabular entries. The calculations were performed on an IBM 650 calculator using the Bell Telephone Laboratories Double-Precision (16-figure) Interpretive System.

In addition to the checks applied by the author, the reviewer collated the values of ber x and bei x with similar data given by Aldis [1] to 21D, for the range $x = 0.1(0.1)6.0$. No discrepancies were detected.

The range, precision, and accuracy of the tables under review establish them as the definitive tables of the Kelvin functions at the present time.

J. W. W.

1. W. STEADMAN ALDIS, "On the numerical computation of the functions $G_0(x)$, $G_1(x)$, and $J_n(x\sqrt{i})$," Roy. Soc. London, *Proc.*, v. 66, 1900, p. 32–43.

10[L].—NUMERICAL COMPUTATION BUREAU, Report No. 11, *Tables of Whittaker Functions (Wave Functions in Coulomb Field) Part 2*, The Tsuneta Yano Memorial Society, 1–9 Yuraku-cho, Chiyoda-Ku, Tokyo, Japan, 1959, 52 p., 26 cm. Price $3.00.

The first part of these tables was reviewed in *MTAC*, v. 12, 1958, p. 86–88. The earlier review contains some errors and fails to give complete information.

The functions considered are defined thus:

$$G_{\xi,l} + iF_{\xi,l} = \exp\left(-\frac{x}{2}\xi\right) \exp\left[-i\left(\frac{l}{2}\pi - \sigma_l\right)\right] W_{i\xi,l+1/2}(-ix).$$

The earlier review has $\frac{1}{2}$ in place of $l + \frac{1}{2}$ in the subscript of the Whittaker function, W, and incorrectly uses $G_{\xi,1/2}(x)$ and $F_{\xi,1/2}(x)$ for the case $l = 0$. Both the previous review and the present tables fail to define $\sigma_l = \arg \Gamma(l + 1 + \frac{1}{2}ix)$.

The table now reviewed is concerned with values of $F_{\xi,0}(x)$ for large ξ. Values of this function are presented to five decimal places, together with δ_r^2 and δ_k^2, where $k = 1/\xi$ and $r = x/(2k)$, corresponding to $r = 0(.1)10$ and $k = 0(.01)1$, that is, $\xi \geqq 1$.

It is observed that for $k = 0$, that is, $\xi \rightarrow \infty$,

$$F_{\xi,0}(x) = \sqrt{2r}J_1(2\sqrt{2r}); \qquad G_{\xi,0}(x) = \sqrt{2r}Y_1(2\sqrt{2r}),$$

which are identifiable as Bessel-Clifford functions multiplied by $2r$. This relation allows comparison with appropriate data in a publication [1] of the National Bureau of Standards; such a comparison has revealed 24 errors (all due to rounding) in the tables under review, thereby suggesting a general accuracy therein within one unit of the fifth decimal.

J. C. P. Miller

The University Mathematical Laboratory
Cambridge, England

1. NBS Applied Mathematics Series, No. 28, *Tables of Bessel-Clifford Functions of Orders Zero and One*, U. S. Government Printing Office, Washington, D. C., 1953.

11[L, M].—W. W. Gerbes, G. E. Reynolds, M. R. Hoes, & C. J. Drane, Jr., *Table of S(x) and its First Eleven Derivatives*, Vol. 1, 2, 3, Air Force Cambridge Research Center, Bedford, Massachusetts, 1958, 27 cm.

The tabulated function $S(x)$ defined by

$$S(x) = \int_0^x \left(\frac{\sin \frac{u}{2}}{u/2}\right)^2 du$$

is related to the sine integral $Si(x)$ by

$$S(x) = 2\left[Si(x) - \frac{1 - \cos x}{x}\right].$$

For ease in computation in the design of antennas, the function $S(x)$ and its first eleven derivatives are tabulated to six decimal places for

$$x = 0°(1°)18{,}000°.$$

The introduction gives the characteristics of the functions, reduction formulas, power series representations, asymptotic expressions, integral representations, differential equations, transforms, addition formulas, etc., and the method of computation.

The tables were computed using the IBM 650 calculator.

Irene A. Stegun

National Bureau of Standards
Washington 25, District of Columbia

12[L, M].—R. HENSMAN & D. P. JENKINS, *Tables of* $\frac{2}{\pi} e^{z^2} \int_z^\infty e^{-t^2}\, dt$ *for Complex z*, Royal Radar Establishment, Malvern, Worchestershire, England. Deposited in UMT file.

The function $\frac{2}{\pi} e^{z^2} \int_z^\infty e^{-t^2}\, dt$ has been tabulated to 6 decimal places for 0(0.02)2.00 in the real part of z and 0(0.02)4.00 in its imaginary part, and also for 0(0.1)10.0 in both real and imaginary parts. Second differences are given and second-difference interpolation in the appropriate table gives 6-decimal accuracy for the whole range covered. The error in using linear interpolation need not exceed a unit in the fourth decimal.

AUTHORS' SUMMARY

13[L, M].—A. V. HERSHEY, *Computing Programs for the Complex Exponential Integral*, NAVORD Report No. 5909, NPG Report No. 1646, U. S. Naval Proving Ground, Dablgren, Virginia, June 1959, iii + 16 p. Figures and tables. 27 cm. Astia Document Service Center, Armed Services Technical Information Agency, Arlington Hall Station, Arlington 12, Va.

In this report there appears a detailed discussion of the use of asymptotic series with remainder in the evaluation of the complex exponential integral on the Naval Ordnance Research Calculator (NORC). Brief descriptions are also given of two NORC subroutines for the calculation of the exponential integral and the sine and cosine integrals of a real argument.

A series expansion of the remainder of the asymptotic series is presented, and is used to evaluate the remainder with improved accuracy.

The author also describes the construction of a rational approximation to the exponential integral that is valid over the negative half of the complex plane outside the unit circle. In appended tables appear approximations to 13S of the coefficients of two polynomials of the fourteenth degree whose quotient gives values of the function $ze^{-z}Ei(z)$ to within a maximum relative error in the absolute value of $2{\cdot}2 \times 10^{-13}$.

The appendix also includes a table of values to 13S of the real and imaginary parts of $Ei(z)$, corresponding to $x = -20(1)20$ and $y = 0(1)20$. These results were obtained on the NORC by means of double-precision arithmetic, using sixteen-point Gauss integration over each unit interval, beginning with $x = -100$, where the value of the integral was considered to be negligibly small.

Comparison by the reviewer of these data with corresponding entries in an extensive earlier set of tables [1], carried to 6D and 10D, revealed only three instances of rounding errors in the latter, all of such size as to lie within the guaranteed limit of a unit in the last decimal place.

J. W. W.

1. NBS Applied Mathematics Series, No. 51, *Tables of the Exponential Integral for Complex Arguments*, U. S. Government Printing Office, Washington, D. C., 1958. See also *MTAC*, v. 13, 1959, p. 57–58.

14[L, M].—K. A. Karpov, *Tablitsy funktsiĭ* $F(z) = \int_0^z e^{x^2} dx$ *v kompleksnoĭ oblasti* (*Tables of the function* $F(z) = \int_0^z e^{x^2} dx$ *in the complex domain*), Izdatel'stvo Akademii Nauk SSSR, Moscow, 1958, 518 p. + 2 inserts, 27 cm. Price 61 rubles.

This is a companion volume to the tables reviewed in *MTAC*, vol. 12, p. 304–305, and completes the tabulation of the error function in the complex plane. The present volume contains 5D or 5S values of the real and imaginary parts of the function

$$F(z) = \int_0^z e^{x^2} dx = u + iv$$

for $z = \rho e^{i\theta}$, $0 \leqq \rho \leqq \rho_0$, $\pi/4 \leqq \theta \leqq \pi/2$ and $\theta = 0$. The quantity ρ_0 depends on θ and decreases from $\rho_0 = 5$ for $\theta = \pi/4$ to $\rho_0 = 3$ for $\theta = \pi/2$. An exception is $\theta = 0$, for which $\rho_0 = 10$. In the introduction, a diagram is given representing the intervals in θ and the value of ρ_0 for each θ, and a table indicates the intervals in ρ in various parts of the volume. As in the earlier volume, the diagram is reproduced on a cardboard inset, which serves also as an index to the numerical tables.

The introduction gives integral representations and series expansions for u and v, graphs of u and v as functions of ρ for selected values of θ, relief diagrams of u and v over the sector of tabulation, a description of the tables and numerical examples showing their use, some useful numerical values, values of cos 2θ, sin 2θ, and values of $(2n + 1)\theta$ in radians for $n = 0(1)5$ and for those values of θ included in the tables. There is also a one-page auxiliary table of $t(1 - t)/4$ for $0 \leqq t \leqq 1$. This table, together with a nomogram for finding $\tilde{\Delta}^2 t(1 - t)/4$, where $\tilde{\Delta}^2$ designates the sum of two consecutive second differences for use in Bessel's interpolation formula, is reproduced also on a cardboard inset.

Using the symmetry properties of $F(z)$, this function can now be evaluated on the real axis and in a sector of half-angle 45° to both sides of the imaginary axis. Between them, Karpov's two volumes contain a very satisfactory tabulation of the error function in the complex plane.

A. Erdélyi

California Institute of Technology
Pasadena, California

15[L, M, S].—Y. Nomura & S. Katsura, "Diffraction of electric waves by circular plate and circular hole," Sci. Rep. Ritu, B-(Elect. Comm.) *10*, No. 1, 1958, 43 p.

The problem of the diffraction of a plane electromagnetic wave by an infinitely thin, perfectly conducting, circular disk of radius a, and the problem of the diffraction of such a wave by a circular hole of radius a in a plane conducting screen are discussed. The method of solution involves the expansion of the two-component Hertz vector in terms of hypergeometric polynomials. The solution is valid for all frequencies. However, convergence is poor when $ka = 2\pi a/\lambda$ becomes large. Tables of values are included of the real and imaginary parts of

$$G_{n\nu}^m = (2m + 4\nu + 1) \int_0^\infty \frac{J_{m+2n+\frac{1}{2}}(\xi) J_{m+2\nu+\frac{1}{2}}(\xi)\, d\xi}{\sqrt{\xi^2 - (ka)^2}}$$

for $m = 0(1)4$, n, $\nu = 0(1)5$, and $ka = 0(.25)5$, where $J_p(\xi)$ is the Bessel function of the first kind of order p. The function $X^m_{\nu s}$ is defined in terms of the equations

$$\sum_{\nu=0}^{\infty} G^m_{\nu n} X^m_{\nu s} = \delta_{sn} .$$

Tables for $X^m_{n\nu}$ for $n = 0(1)4$, $\nu = 0(1)2, 5$, $m = 0(1)4$, over the same range of ξ, are also included.

In addition, tables are given which are useful for the calculation of the field distribution at large distances, and tables are given which enable one to determine the current distribution on the plate and the electric field distribution on the whole.

All table entries are given to four decimal places. However, no indication is given as to the numerical method of evaluating the table entries nor as to their actual accuracy.

A. H. T.

16[P].—Charles J. Thorne, *Temperature Tables: Part 1. One-Layer Plate, One-Space Variable, Linear*, NAVORD Report 5562, U. S. Naval Ordnance Test Station, California, 1957, iv + 711 p., 28 cm.

This table is concerned with listing the solution of the heat conduction equation in a plate of finite thickness, with heat transfer at both faces. In mathematical form

$$\begin{array}{lll} U_{XX} = kU_t & 0 < X < L, & t > 0 \\ KU_X = -h_i(U_i - U) & X = 0, & t > 0 \\ KU_X = -h_0(U - U_0) & X = L, & t > 0 \\ U = U_0 & t = 0, & 0 < X < L \end{array}$$

where the conductivity K, density ρ, specific heat c, diffusivity $h = c\rho/K$, heat transfer coefficients h_i and h_0, and stagnation temperatures U_i and U_0 are assumed to be constant. L is the plate thickness, x is the distance, and t is the time. For the tables a dimensionless system of variables is adopted, i.e., $x = X/L$, $kL^2\tau = t$, $u = (U - U_0)(U_i - U_0)$, $\alpha_0 = h_0L/K$, $\alpha_i = h_iL/K$. Then the above problem becomes

$$\begin{array}{lll} u_{xx} = u_\tau & 0 < x < 1, & \tau > 0 \\ u_x = -\alpha_i(1 - u) & x = 0, & \tau > 0 \\ u_x = -\alpha_0 u & x = 1, & \tau > 0 \\ u = 0 & \tau = 0, & 0 < x < 1 \end{array}$$

The analytical solution of this problem is

$$u(x, \tau) = 1 - \frac{\alpha_0(1 + \alpha_i x)}{\alpha_0 + \alpha_i + \alpha_0\alpha_i} + 2 \sum_{n=1}^{\infty} \frac{e^{-\beta_n^2\tau}}{\beta_n D'(\beta_n)}$$

$$\cdot \{\alpha_i[\beta_n \sin \beta_n - \alpha_0 \cos \beta_n] \sin \beta_n x + \alpha_i[\beta_n \cos \beta_n + \alpha_0 \sin \beta_n] \cos \beta_n x\},$$

where

$$D'(\beta) = -\beta(2 + \alpha_0 + \alpha_i) \sin \beta + (-\beta^2 + \alpha_0 + \alpha_i + \alpha_0\alpha_i) \cos \beta,$$

and the β_n's are the positive roots of

$$(\alpha_0\alpha_i - \beta^2) \sin \beta + \beta(\alpha_0 + \alpha_i) \cos \beta = 0.$$

The table gives the dimensionless temperature u, where $0 < u < 1$, for the following ranges of parameters

x = 0(.01).02(.03).05(.05).3(.1).7(.05).95(.03).98(.01)1

τ = .001(.0005).002(.001).008(.002).01(.01).08(.02).1(.1).8(.2)1(1)8(2)10(10)80(20)100(100)800(200)1000

α_i = .001, .002, .004, .006, .01, .02, .04, .06, .1, .2, .4, .6, 1, 2, 4, 6, 10, 20, 40, 60, 100, 200, 400, 600, 1000.

α_0 = 0, .001, .004, .01, .04, .1, .4, 1, 4, 10, 40, 100, 400, 1000.

The tabular entries are given to five figures, with better than three of the figures being accurate. For the most part, the error appears to be one or two units in the fifth figure.

The table should be very useful to those people who are engaged in design work involving heat transfer, as, for example, rocket nozzle design. The introduction also contains a generalization of the heat conduction problem defined above which can be solved by means of the tables.

The tabular entries are printed with reasonable clarity. There are, however, a few obvious misprints in the introduction.

R. C. ROBERTS

Naval Ordnance Laboratory
Silver Spring, Maryland

17[S].—G. A. BARTHOLOMEW & L. A. HIGGS, *Compilation of Thermal Neutron Capture Gamma Rays*, Report AECL No. 669, 1958, 146 p., 27 cm. Available from Scientific Document Distribution Office, Atomic Energy of Canada Limited, Chalk River, Ontario, Canada. Price $2.50.

This report presents a compilation of energy, absolute intensities, and spectral distribution of gamma rays produced by capture of thermal neutrons, together with a complete bibliography of information on this subject through June 1, 1958. The results obtained from measurements at the Chalk River Laboratories over several years using the pair spectrometer have been reviewed. These results have been modified where necessary such that all intensity determinations are presented on a uniform basis. The accuracy of pair spectrometer intensity measurements is discussed.

Included in the tables are the energies and intensities (photons per hundred captures) of resolved gamma rays obtained from experiments in which absolute intensities were determined. References to other data not tabulated is also given. Where an appreciable portion of the gamma ray spectrum is unresolved, a spectral distribution curve is given. Most of the curves plot the number of gamma rays per capture per Mev as a function of energy. Results published by the Moscow group are included. A rough measure of accuracy and completeness of the results is given with each tabulation and with most of the curves.

K. SHURE

Westinghouse Electric Corporation
Bettis Atomic Power Division
Pittsburgh, Pennsylvania

18[S].—M. E. Rose, *Internal Conversion Coefficients*, Interscience Publishers, Inc., New York, 1958, xxii + 173 p., 31 cm. Price $6.25.

When a nucleus is in an excited state for which the excitation energy is insufficient for the emission of nuclear particles, the de-excitation will proceed predominantly by either one of two competing mechanisms. Either a γ-ray will be emitted or the nuclear excitation will be transferred to one of the orbital electrons resulting in the ejection from the atom of this electron. The latter process is referred to as internal conversion. If N_e is the number of conversion electrons emitted per second and N_q is the number of photons emitted per second, the internal conversion coefficient α is defined as

$$\alpha = N_e/N_q .$$

This book gives a comprehensive account of a ten-year program devoted to the calculation of internal conversion coefficients. It contains a ten-page introduction which gives a precise and thorough account of the physical and numerical approximations made in the course of the calculations of the tables. These in turn constitute the bulk of the work, 164 pages.

The conversion coefficients are strongly dependent on k, where kmc^2 is the transition energy, Z the atomic number, L the angular momentum change, and on $\Delta\pi$, the parity change. The tables list values of α_L and β_L, $(L = 1, 2, 3, 4, 5)$, the coefficients for 2^L electric pole and 2^L magnetic pole conversions, respectively, for $k = 0.05(0.05)0.2(0.2)1.0(0.5)2.0$ and $Z = 25(1)95$.

Also included in the tables are values of certain radial matrix elements $R_k(m)$ and $R_k(e)$ for the K shell. These are uncorrected for screening and for finite nuclear size.

The author lists the sources of error in determination of the radial wave functions, which are a fundamental set of intermediate quantities in the calculation of the tables, and expresses the view that all of these errors are small and amount at most to 1–3 per cent. He expresses the view that the irreducible minimum error involved in any calculation of internal conversion coefficients, aside from nuclear structure effects, is just smaller than the experimental error in the best measurements now available.

The author states that, "interpolation in the tables will be necessary only in the energy variable k. For this purpose interpolation on a log α or log β versus log k is advisable since the plots of the conversion coefficients on a log-log scale show very little curvature."

A. H. T.

19[S].—C. H. Westcott, *Effective Cross Section Values for Well-Moderated Thermal Reactor Spectra*, Report AECL No. 670, 1958, 27 p., 27 cm. Available from Scientific Document Distribution Office, Atomic Energy of Canada Limited, Chalk River, Ontario, Canada. Price $1.00.

This report is concerned with the determination of an effective neutron absorption cross section, which cross section is recommended for calculations involving reaction rates. The effective cross section is defined in terms of a neutron density distribution per unit velocity. The neutron spectrum assumed consists of a Maxwellian distribution at a temperature T°K plus an admixture of a $1/E$ distribution

of flux per unit energy interval, the admixture being controlled by an epithermal index r. For $r = 0$, the spectrum is a pure Maxwellian.

Using this spectrum, the effective cross section, σ can be written as

$$\sigma = \sigma_{2200}(g + rs)$$

where σ_{2200} is the microscopic absorption cross section at 2200 m/sec. The g and s factors depend on the shape of the absorption cross section as a function of neutron energy. Specifically for nuclides obeying the $1/v$ law over the entire energy range, $g = 1$ and $s = 0$.

The accuracy of the results obtained depends on the input data used which, in general, has been taken from the 1958 revision of the Brookhaven *Neutron Cross Section Compilation*. Tables of σ and g and s factors in 20 centigrade degree steps from 20°C to 760°C are listed for $r = 0.03$ or $r = 0.07$. These values of r correspond to average parameters appropriate for the moderator and fuel rods respectively of the NRX Reactor. In some instances σ for $r = 0$ are given. Elements which follow the $1/v$ law fairly closely in the thermal region do not usually have g values listed. The applicability of the compilation is limited to well moderated reactors and to thin samples in which self shielding has been neglected.

K. Shure

Westinghouse Electric Corporation
Bettis Atomic Power Division
Pittsburgh, Pennsylvania

20[T].—Alvin Glassner, *The Thermochemical Properties of the Oxides, Fluorides, and Chlorides to 2,500° K*, Argonne National Laboratory ANL-5750, U. S. Government Printing Office, Washington, D. C., 1957, vi + 74 p., 27 cm. Price $.45.

This table gives empirical equations for the thermodynamic properties per mole (heat capacity Cp, enthalpy H, entropy S, and gives free energy F) in the form of power series in the absolute temperature T. The heat capacity at constant pressure is fitted to the equation:

$$C_p = a + (b \times 10^{-3})T + (c \times 10^{-6})T^2 + \frac{d \times 10^5}{T^2}$$

Only three parameters are evaluated; either c or d is set equal to zero. Integration of the heat capacity to give H, S, and F requires two additional constants of integration, A and B. The coefficients are tabulated for solid, liquid, and gaseous states of the elements (Table I—3 p.), the oxides (Table II—4 p.), the fluorides (Table III—5 p.) and the chlorides (Table IV—5 p.). Each of these tables includes, in addition, the heat and entropy of the phase transitions, the entropy at 298°K, and appropriate references to the source material. Tables V, VI, and VII give the enthalpy and free energies of formation of each substance from the elements at 298°K, as well as the coefficients Δa, Δb, Δc, Δd, ΔA, and ΔB needed to calculate values at other temperatures.

The publication concludes with 21 pages of graphs showing the temperature dependence of ΔF_f from 300° to 2500°K.

Robert L. Scott

University of California
Los Angeles, California

21[W].—Kenneth J. Arrow, Samuel Karlin & Herbert Scarf, *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, Stanford, California, 1958, x + 340 p., 25 cm. Price $8.75.

This work is the initial volume in the Stanford Mathematical Studies in the Social Sciences. The reviewer joins the principal authors in recommending careful attention to the first two chapters: in Chapter I, Arrow presents a remarkably concise and enlightening discussion which, more constructively than anything else the reviewer has read, relates inventory theory to economics; in Chapter II, this useful survey is continued as the principal authors treat common features of many inventory models after placing them within a realistic framework for decision models. The Introduction ends with summaries of results of the remaining three parts: Optimal Policies in Deterministic Inventory Processes, Optimal Policies in Stochastic Inventory Processes, and Operating Characteristics of Inventory Policies. This book is judged to devote reasonable attention to computing problems both for calculation of solutions and for illumination. Reading of individual chapters has convinced the reviewer that the general promises on computing made by the authors on pages 16–19 were honestly kept. The frequent graphs and tables are uniformly helpful and pleasing. A bibliography of four pages covering mainly the years 1955–1957 is also included. It is to be hoped that subsequent volumes of this Stanford Series will push forward into the wide reaches of inventory problems including, for example, areas of demand prediction, measures of utility for satisfying differing demand patterns, and even seemingly prosaic questions such as how to maintain records of extensive inventory systems. In summary, this book is a substantial contribution to the mathematics of inventory and production problems. Since it provides a sound exposition over quite a broad range, it should serve as a valuable source for further research.

W. H. Marlow

The George Washington University
Washington, District of Columbia

22[W, Z].—Daniel D. McCracken, Harold Weiss, & Tsai-Hwa Lee, *Programming Business Computers*, John Wiley & Sons, Inc., New York, 1959, xvii + 510 p., 24 cm. Price $10.25.

Here is a well-written book about the elements of programming high-speed electronic computers. It is particularly written for those people who are interested in the programming of management data problems. The book touches upon down-to-earth details such as verifying the program accuracy, input and output programming, and rerun techniques. It discusses the advantages and disadvantages of machine-aided coding. In general, *Programming Business Computers* is a comprehensive survey of programming with special emphasis on business data processing.

Arthur Shapiro

Applied Mathematics Laboratory
David Taylor Model Basin
Washington, District of Columbia

23[X].—J. N. Goodier & P. G. Hodge, Jr., *Elasticity and Plasticity*, v. 1, Surveys in Applied Math. Series, John Wiley & Sons, Inc., New York, 1958, ix + 152 p., 23 cm. Price $6.25.

The articles of this series are said to be "aimed at a broad, mathematically literate audience looking for an up-to-date account of modern progress in applied mathematics and an appraisal of future promising research directions." The first author of Volume 1, J. N. Goodier, who contributed the article entitled, "The Mathematical Theory of Elasticity," assumed that the reader was well versed in the theory of elasticity, and made no effort to make his article self-contained. He contented himself with a brief survey of "those significant recent developments believed least known to readers whose first language is English." However, even this intention is not fully carried out and a list of topics omitted is given at the end of the article. Three pages of bibliography are given, which include only those books and papers actually discussed or cited.

The major portion of the discussion deals with work of Russian authors. Great stress is given to the work of Muskhelishvili and to investigations inspired by it.

There is no mention in this article of the application of numerical methods to problems in elasticity, aside from a reference to the survey of numerical methods in conformal mapping given by G. Birkhoff, D. M. Young and H. Zarantonello, in *Proc. Symp. Appl. Math*, v. 4, 1953, p. 117.

The second article written by Phillip G. Hodge and entitled, "The Mathematical Theory of Plasticity," is practically self-contained and satisfies, very well, the needs of the member of the audience described in the first paragraph of this review.

Chapters 1 to 4 of this article are theoretical in nature, and Chapters 5 to 7 are concerned with applications of the theory. In particular, Chapter 5 is a well written, moderately exhaustive treatment of the behavior of a simply supported circular plate under a uniform normal pressure. In Chapter 6 other problems are discussed more briefly in an attempt to illustrate the current state of development in plasticity problems.

Particular attention is paid to significant Russian contributions. Chapter 7 contains transcription of parts of a report by W. Prager on Russian contributions up to 1949, and a section by the author entitled, "Contributions from 1949 to 1955."

There is no mention made in this article of the applications of numerical methods to problems in plasticity.

A. H. T.

24[X].—L. V. Kantorovich & V. I. Krylov, *Approximate Methods of Higher Analysis*, Translated from the third Russian edition by Curtis D. Benster, Interscience Publishers, Inc., New York, 1958, xv + 681 p., 24 cm. Price $17.00.

In the April 30, 1959 issue of *Le Monde*, on page 5, there is a description of the organization of scientific activities in Russia, in the course of which the following remark is made: "Contrairement aux Américains, les Russes paraissent parfaitement au courant de la littérature mondiale." The author is Maurice Letort, "président du comité consultatif de la recherche scientifique et technique."

One would like to be indignant, but unfortunately the gibe is deserved. In fact, many Americans who visit Russia, or otherwise make contacts with Russian scientists, are amazed at how up-to-date their acquaintance is with American literature, which implies that their own acquaintance with Russian literature is much less so. However, the article in *Le Monde* also provides a partial explanation by describing the extensive Russian facilities for translating and abstracting (2000 full time

"collaborators," plus many part time). It is hardly necessary to comment on the meagerness of our own facilities.

Until recently, even reviews and abstracts of Russian literature were pitifully sparse, although here there has been vast improvement. The original Kantorovich-Krylov has been known and appreciated by a few Americans, probably largely due to informal publicity given it by George Forsythe, who encouraged the making and publishing of the present translation. But *Mathematical Reviews* has no review of the second edition, published in 1941, and for the third edition it listed chapter headings and remarked, in an unsigned article, only that the edition differed very little from the previous one.

At any rate, we can be grateful to translator and publisher for the present volume. The book itself is concerned mainly with the numerical solution of partial differential equations, as the title to the first edition (1936) indicated. The first chapter deals with expansion in series, both orthogonal and nonorthogonal, with a section on the improvement of convergence. Next come methods of solution of Fredholm integral equations with applications to the Dirichlet problem. Then comes a chapter on difference methods, and one on variational methods. This accounts for slightly more than half of the book. There follow two chapters, for a total of nearly 250 pages, on conformal methods, and finally about 50 pages on Schwarz's method. Throughout, the presentation is extremely readable, with the inclusion of numerous examples, but no exercises. Unfortunately there is no index, either, although the table of contents is fairly detailed (5 pages).

In organization the translation deviates from the original only in collecting the references at the end, with footnotes referring to author and number. This I consider to be desirable. In detail the translation is faithful and quite clear. At times the phraseology is too faithful for elegance, and on rare occasions the translator is even led astray. One such example occurs on page 7: "Just as there, we may separate the problem into two, and moreover in each case the conditions are null on two sides." While the reader should understand what is meant, there are two faults to find here. First, "pričem" should be translated as "where," not "and moreover." Second, a condition cannot be null. I confess, I do not understand the construction in the original, which is "usloviya nulevye," and perhaps the translator can be forgiven for assuming the adjective to be in predicate form in spite of the ending. Perhaps the authors themselves were careless.

But one could always find fault with details, whereas the important thing is that the book is now available to readers of English. Again our thanks to publishers and translator.

A. S. Householder

Oak Ridge National Laboratory
Oak Ridge, Tennessee

25[Z].—Franz L. Alt, *Electronic Digital Computers*, Academic Press, New York, 1958, x + 336 p., 23 cm. Price $10.00.

In the preface to his book, Alt addresses himself primarily to "physicists, chemists, engineers and others in similar occupations who have occasion to require the solution of computational problems by means of digital computing machines."

Alt's primary motivation with respect to this audience is to provide sufficient introductory information to improve communication between the "originator of a computing problem and the team around the machine."

The Introduction, Part I, leads the reader gracefully toward Alt's objectives with a well formulated statement of the stages through which a problem goes on its way to the number factory. Having outlined the required process of problem formulation to problem analysis to programming to coding to machine computation, the author proceeds to discuss the process in reverse. Thus, Part II provides a functional survey of automatic digital computers and Part III discusses programming and coding. By the time the object audience reaches more familiar ground in Part IV (Problem Analysis) and the statement of computer applications in Part V, it has gained the hindsight necessary to make modern computing methods more palatable.

The historical survey of automatic digital computers in Part II is somewhat weakened by the selection of the memory type as a principal classification characteristic. The more significant factor of operation time, or conceptual factors such as the stored program are therefore undermined. However, the bulk of Part II adequately introduces the key factors differentiating one machine from another, that is, number representation and memory, arithmetic, control and input-output organs. The reviewer felt a lack of graphical presentation of the material in Part II—the inclusion of so much data in the same form as the normal prose is likely to lose the reader for whom this book is written.

Part III remains consistent in its reverse discussion of programming and coding, by describing coding first and then programming. A 4-address instruction set is defined and the coding of a simple arithmetic expression and trigonometric function is used as a vehicle for explaining coding operations. Single address coding is also described in the terminology of IBM manuals. The sections on programming demonstrate the use of flow charts and define factors and terminology significant to computing tactics.

Part IV, covering problem analysis, is by far the strongest part of the book. It collects, in very readable form, methods of computation used in numerical solution of ordinary differential, partial differential, and algebraic equations. As mentioned in the preface, there is no attempt at rigor in this presentation. The discussion of numerical methods is well seasoned with qualitative comments reflecting computational experience and with references to the 210 papers listed in the appended bibliography.

This book cannot by itself serve as a text for the classroom. This book will not serve as a reference textbook in the sense of its cataloging completeness. However, it is this reviewer's feeling that *Electronic Digital Computers* will be found on reference shelves for many years, by virtue of its very readable presentation of Alt's extensive experience in high-speed computation. The goal, established in the preface, is very adequately achieved.

GERALD ESTRIN

University of California
Los Angeles, California

26[Z].—James T. Culbertson, *Mathematics and Logic for Digital Devices*, D. van Nostrand Co., New Jersey, 1958, x + 224 p., 23 cm. Price $4.85.

This text is intended for students with a background of college algebra who plan to enter the computer field. As the author points out in his first sentences, it "presents neither a course in programming computers nor a mathematical analysis of computing mechanisms. Preliminary to these things, it provides a course in some mathematics which the student will find useful later". Specifically, the material covered includes some combinatorics and probability, Boolean algebra and propositional calculus, with applications to switching circuits. These topics are covered by a text written in a highly readable colloquial style, plentifully illustrated by examples and by the tremendous total of 1262 numbered exercises. (The latter are "dressed-up", provided with continuity and some attempt at humor in a manner which the reviewer finds slightly repellent—but this is a matter of taste. The device does have the merit of permitting problems which are essentially identical to appear in radically different formulations.)

The unifying concept of the first part of the text is that of the neuron model. Various successive refinements of the receptor-central-effector system are introduced, leading to adders and other complicated input-output systems. Chapter I is introductory, presenting the summation and product notations and the ideas of an algorithm and iterative approximation. Unfortunately, two of the five examples of the summation notation are incorrect, and there is no clear distinction between an algorithm and an iteration. After correctly defining the former as concluding in a finite number of steps, the author introduces "Newton's algorithm" for the square root, which is an example of the iteration process. Further, he states the completely false result that $B = \frac{1}{2}(A + N/A)$ is always a better approximation to $\sqrt{N}$ than A. These points will illustrate that statements in the text must be carefully watched; accuracy has frequently been sacrificed to simplicity of statement.

Chapters II and III present the basic facts about combinations, permutations, and elementary frequency probability. Chapter IV is an excellent elementary account of arithmetic in various radix systems and of conversion from one system to another. The various mechanical procedures, such as subtraction by complementation and division by subtraction, are considered in detail. Except for minor matters of choice of language, this chapter may be highly recommended.

The second portion of the book deals with logic. The larger part of Chapter V is an exposition of the syllogistic logic in its full mediaeval pattern, including even the vowel notation of the scholastics, and omitting only the traditional mnemonics, Barbara, Darii, etc. The reviewer finds this portion of the book utterly astounding. It is as if one were to come across a long commentary on *De Rarum Natura* in a text on modern atomic physics. Neither mathematicians nor computer engineers use syllogisms; what purpose can this chapter serve? The latter portion discusses relations, omitting reflexivity and giving much more stringent definitions of one-many and many-one than customary.

The final three chapters develop successively Boolean algebra, the propositional calculus, and the model of the latter in terms of switching circuits. The reduction to normal form and initial simplification are well presented. Too much reliance is placed on checking by means of Venn diagrams. The student is not adequately

warned that, while a Venn diagram illustration of Boolean inequalities is always available, equality in a particular Venn diagram does not necessarily imply general equality.

In summary, the text covers matters of algebra, arithmetic, and logic that students should know before taking advanced courses in logical programming or component design. It may be recommended as a text or for collateral reading if the instructor will warn the student of possible pitfalls and inaccuracies.

J. D. Swift

University of California
Los Angeles, California

27[Z].—Dept. of the Army, *Catalog of Commercially Available Automatic Data Processing Systems*, Department of the Army Pamphlet No. 1–250–4, 1958, 107 p., 26 cm.

This pamphlet is an excellent compilation and description of automatic data processing systems which are commercially available as of July, 1959. It contains the descriptions of twenty-five digital data processing systems ranging in size from the Bendix G15D and the Royal McBee LGP30 to the UNIVAC 1105 and the DATAmatic 1000. A photograph is included with each computer description.

This compilation is subdivided into categories entitled, respectively, General Description, System Components, Programming, Personnel Requirements, and Site Preparation.

The systems component category describes a typical computer configuration consisting of central processor, arithmetic unit, input-output control, high speed memory, magnetic tape units, paper tape units, card readers and punches, and high speed printers. This category lists some pertinent characteristics such as word length, numeric characters per word, timing, pulse repetition rate, size of memory, checking, and error correcting features. A rather complete list of specifications and characteristics of input and output media is included. The instruction word structure is also presented.

The personnel requirements category recommends a programming and operating complement of personnel but excludes maintenance personnel. Manufacturer's training of operators and programmers is discussed, with an option of training at the manufacturer's premises or at the installation site.

The over-all floor space, floor loading, and air conditioning requirements are also given in the site preparation category.

However, the major contributions of this pamphlet are the tables of cost, power requirements, and physical characteristics of each unit. These tables also present rental, purchase, and maintenance costs.

This reviewer considers this pamphlet an excellent guide to all computer users who require a ready reference in the field.

Alexander C. Rosenberg

Applied Mathematics Laboratory
David Taylor Model Basin
Washington, District of Columbia

28[Z].—Martin Gardner, *Logic Machines and Diagrams*, McGraw-Hill Book Company, Inc., New York, 1958, ix + 157 p., 23 cm. Price $5.00.

Only on rare occasions does one come across a book written on a technical subject which is entertaining as well as informative. It is also unusual to find a book on the mechanistic aspects of formal logic which does not begin with either Venn diagrams or a description of Boolean algebra. The author has presented an historical survey of the subject in a somewhat narrative fashion, beginning with an almost complete biography of Ramon Lull and ending with speculations on the future of logic machines. The "References" after each chapter are considerably more than just references, and make as interesting reading as the text. The book is by no means devoid of the author's opinions and no attempt has been made at concealment. In fact, it is amusing to note that, even though some of the artifacts and methods described in the book are treated somewhat modestly by the author, he cannot resist the temptation to devote a chapter to a method which he, himself, has devised. This, I am sure, is understandable to any person who has worked in the field.

Persons interested in the field of logic, either as a subdivision of philosophy or as an aid to digital computer design, will find the book well worth its reading time. My only complaint is that he has not included the work done in the area of computer design, which could be interpreted as legitimate subject matter under this title.

Nelson T. Grisamore

The George Washington University
Washington, District of Columbia

TABLE ERRATUM

273.—E. JAHNKE & F. EMDE, *Tables of Functions with Formulae and Curves*. 4th edition, Dover Publications, New York, 1945.

On p. 99, the caption for Fig. 55 includes values of ω, ω', k, k', e_1, e_2, e_3, g_2 and g_3 associated with the Weierstrassian $\wp$ function. It is, of course, impossible to tell with certainty which of these numbers were intended to be assumed as given exactly. A reasonable inference, however, is that $k = 0.8$ and $e_1 - e_3 = 1$ were intended as given. If this be the case, the values of e_3 and g_3 are in error. The 3D values of e_3 and g_3 should, then, be -0.547 and -0.093 respectively (indeed, $e_3 = -0.5467$ and $g_3 = -0.09252$, to 4S).

THOMAS H. SOUTHARD

University of California
Los Angeles, California

EDITORIAL NOTE: Under the assumptions stated in the preceding communication, the values of ω and ω' are, respectively, 1.995303 and 1.750754i to 6D, which when rounded to two decimal places appear to be *exact* numbers, as one might erroneously infer from the values shown to that accuracy in the caption under discussion.

J. W. W.

CORRIGENDA

DANIEL C. FIELDER, "A note on summation formulas of powers of roots," *MTAC*, v. 12, 1958, p. 197.

The term $27a_1a_2^2a_4/a_0^4$ has been omitted from the expression for S_9. The fourth line in this expression should accordingly be corrected to read

$$+ (9a_1^2a_6 + 27a_1^2a_2a_5 + 27a_1^2a_3a_4 + 27a_1a_2a_3^2 + 27a_1a_2^2a_4 + 9a_2^2a_3)/a_0^4.$$

G. P. M. HESELDEN

HER MAJESTY'S NAUTICAL ALMANAC OFFICE, *Subtabulation, A Companion Booklet to Interpolation and Allied Tables*, *MTAC*, Review **18**, v. 13, 1959, p. 127–129.

	for	read
p. 128, first displayed formula	$_p\delta^3_1$	$p\delta_1$
p. 128, last displayed formula	$\frac{1}{6}vr$	$\frac{1}{6}r$
	$B_3(rv)C$	$B_3(rv)c$
	$\gamma_0^2 + \gamma_1^7$	$\gamma_0^4 + \gamma_1^4$
p. 129, first displayed formula	$\frac{1}{2}$	δ_4

JOHN TODD

The SIAM Review

VOLUME 2 — January, 1960 — NUMBER 1

ARTICLES

Report of the President for 1959 .... *D. L. Thomsen, Jr.*
Professional mathematicians in government and industry .... *R. E. Gaskell*
Linear and nonlinear equations; local and global properties .. *F. A. Ficken*
Some applications of the pseudoinverse of a matrix ..... *T. N. E. Greville*
On infinite integrals containing products of Bessell functions *Joel L. Ekstrom*
A note on smooth interpolation .... *T. J. Rivlin*
Fitting position data to minimize velocity errors .... *David S. Stoller*
Characteristic root bounds of Gerschgorin type .... *A. B. Farnell*

PROBLEMS

Steady-state diffusion-convection .... *G. F. H. Gardner*
On a binomial identity arising from a sorting problem ...... *Paul Brock*
A center of gravity perturbation .... *M. S. Klamkin*

SOLUTIONS

N-dimensional volume (Eisenstein and Klamkin) .... *I. J. Schoenberg*

BOOK REVIEWS

Handbook of Automation, Computation, and Control (Grabbe, Ramo, and Wooldridge) .... T. F. Green
Analysis of Straight Line Data (Acton) .... Philip R. Monson
On Numerical Approximation (Langer, ed.) .... Ramon E. Moore
Ordinary Differential Equations (Kaplan) .... Seymour Schuster
Automation and Computing (Booth) .... Maria Schniewind
An Introduction to Combinatorial Analysis (Riordan) ...... S. Kullback
Analysis of Industrial Operations (Bowman and Fetter) ... Jack C. Rogers
Readings in Linear Programming (Vajda) .... Clement Winston

NEWS AND NOTES

W

BER 1